SOLID GEOMETRY.

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SOLID GEOMETRY.

BY

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PREFACE.

It was with a feeling of great discouragement that I began the preparation of another Edition of this work, deprived, as I was, of the valuable assistance of my friend Mr. Wolstenholme, in working with whom I had had so much pleasure while writing the First Edition. Mr. Wolstenholme, who is now Professor of Mathematics in the Royal Indian Engineering College at Cooper's Hill, thought that there would be great difficulty in carrying on this work satisfactorily by correspondence, even if the important duties in which he is engaged did not fully occupy his time; I was, therefore, reluctantly obliged to undertake the whole labour of remodelling our original work.

As we contemplated making additions, and many alterations both in form and substance, my friend desired that his name might not appear in the Second Edition, and I have been compelled to alter the title of the work, and to take the responsibility of the changes which have been introduced.

The problems which appeared in the former Edition were for the most part original, and a large proportion of them were due to Mr. Wolstenholme; in this department, therefore, a most important one in my opinion, I have not lost the advantage of his valuable assistance.

The present Edition is intended, in its complete form, to occupy two volumes, but for the convenience of Students who may wish to have in one volume all those portions of Solid Geometry which would be useful to them in their studies of Physical subjects, I have endeavoured, as far as I could without material departure from the arrangement which I considered best for the proper treatment of the subject, to include in the first volume nearly all that will be required from their point of view; with this object, I have reserved for the second volume those parts which are chiefly interesting as Pure Geometry.

The Student who desires to confine his reading to the more practical portions of the subject should omit Chapters VI., VII., VIII., IX., Art. 155—157, XV. and XVII., the Three-Plane system of Coordinates being employed exclusively in the remaining chapters.

I feel bound to say a few words with respect to my persistence in retaining the word 'Conicoid'

to represent the locus for the equation of the second degree. It was natural that the distinguished analyst, who has done so much towards the investigation of the properties of surfaces of higher degrees than the second, should seek a term for that of the second degree, which would connect it with those of higher degrees. But I cannot help thinking it unfortunate that the terms 'quadric' should have been selected, which had already a different meaning. I quote the words of the author of the well-known treatise on Higher Algebra: "It "is convenient to have a word to denote the "function itself without being obliged to speak of "the equation got by putting the function = 0. "The term 'quantic' denotes, after Mr. Cayley, "a homogeneous function in general, using the "words 'quadric,' 'cubic,' 'quartic,' 'n-ic,' to "denote quantics of the 2nd, 3rd, 4th, nth, degrees." Now, 'quadric,' as used in the other sense, is not even the equation found, but it takes two steps and becomes the locus of the equation.

I consider that the surface of the second degree at present, whatever may be the case in some future development, stands on a platform of its own, on account of the services which it has rendered to all departments of Mathematical Science, and well deserves a distinctive name instead of being recog-

nised only by its number, a mode of designation which, I am informed, a convict feels so acutely. Man might be always called a biped, because besides himself there exist a quadruped, an octopus, and a centipede, but, on account of his superiority, it is more complimentary to call him by some special name.

The useful word 'conic' being well-established, the term 'conicoid' seems to suggest all that can be required, when it is employed to designate the locus of the equation of the second degree in three dimensions, at least so long as the analogous words spheroid, ellipsoid, and hyperboloid are in use, at all events it is not open to the great objection of being equally applicable to plane curves, as is the term 'quadric;' cubics and quartics being actually so employed in Salmon's *Higher Plane Curves*, Chapters V. and VI.

To the many excellent mathematicians, whose talent is shewn in the composition of the yearly College papers and the papers set for the Mathematical Tripos examination, I am indebted in the highest degree both for the problems which I have added to the collection, and also for the hints derived from them in the treatment of the subject itself.

I have also to make thankful acknowledgments for

the valuable assistance received from my friends. Mr. Moulton, of Christ's College, has given me great help in parts of the subject which, except in the chapters on the general equation, do not appear in this volume. Mr. H. M. Taylor, Fellow of Trinity College, was kind enough to look over many of the proof sheets; and I am indebted to Mr. Ritchie and Mr. Main, of Trinity College, and Mr. Stearn, of King's College, for their kindness in testing a large number of the problems, as well as in looking over the proof sheets throughout the process of publication. But especially I wish to thank Mr. Stearn for the great assistance which he has rendered in superintending the work during my frequent absence from Cambridge, and also for his many valuable criticisms.

CAMBRIDGE,
October, 1875.

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GEOMETRY OF THREE DIMENSIONS.

CHAPTER I.

ON COORDINATE SYSTEMS.

1. BEFORE entering upon the application of Algebra to the investigation of Theorems, and to the solution of Problems, in Solid Geometry, we shall premise on the part of the student a complete knowledge of all the ordinary processes adopted in the case of Plane Geometry.

By this means we shall avoid the necessity of entering upon the subject of the interpretation of the affection denoted by the sign $(-)$ prefixed to a symbol; since it is known that, if $+a$ denote a line of length a measured in any direction from a point in a line straight or curved, $-a$ may be interpreted to denote a line of length a measured in the opposite direction from any other point in the line, without this hypothesis involving any infringement of the laws of combination of these signs, assumed as the fundamental laws of Symbolical Algebra.

2. Our first object will be to explain how the position of a point in space can be represented by algebraical symbols, and with this view we shall describe two of the different coordinate systems which it has been found convenient to adopt; reserving the consideration of other systems for future chapters, when the student shall have acquired some familiarity with the subject. And it will be found that each of the systems has its peculiar advantage, according to the nature of the theorem or problem which is the subject of examination.

Coordinate System of Three Planes.

3. In the coordinate system of three planes, three planes xOy, yOz, zOx are fixed upon as planes of reference, which may be either perpendicular to one another, or inclined at angles which are known.

The three lines in which they intersect are called *coordinate axes*, and the point in which they meet the *origin of coordinates*.

The position of a point in space is then completely determined, when its distance from each of the planes, estimated parallel to the coordinate axes, and the direction in which those distances are measured, are given.

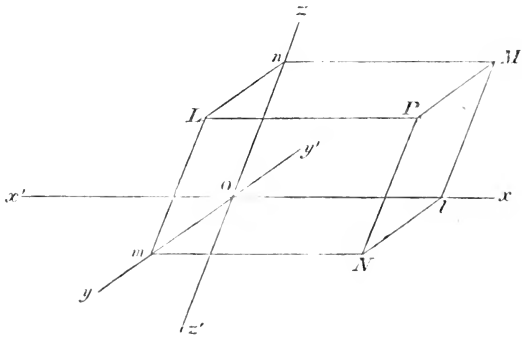
The absolute distance and the direction of measurement are included in the term *algebraical distance*.

Thus $+a$ and $-a$ are the algebraical distances of two points whose absolute distances from the plane yOz are each a, and which are measured, the first in the direction Ox, the second in the direction xO from that plane.

These algebraical distances are called the *coordinates* of a point in this system, and are usually denoted by the letters x, y, and z.

The point, of which these are coordinates, is described as the point (x, y, z).

Produce xO, yO, zO backwards to x', y', z'; then, if a, b, c



are absolute lengths, (a, b, c) denotes a point in the compartment $xyzO$, $(-a, b, c)$ in $x'yzO$, $(a, -b, c)$ in $xy'zO$, $(a, b, -c)$ in $xyz'O$, $(a, -b, -c)$ in $xy'z'O$, $(-a, b, -c)$ in $x'yz'O$, $(-a, -b, c)$ in $x'y'zO$, $(-a, -b, -c)$ in $x'y'z'O$.

4. If a parallelepiped be constructed, whose faces are parallel to the coordinate planes, the point $P(a, b, c)$ being the other extremity of the diagonal drawn from the origin, the edges LP, MP, NP will be the coordinates of the point P supposed in the compartment $xyzO$.

Also, it is obvious that $x = a$ for every point in the plane face $PNlM$, or that $x = a$ is the equation of that plane, as $y = b$ and $z = c$ are the equations of the planes $PLmN$ and $PMnL$ indefinitely extended in every direction.

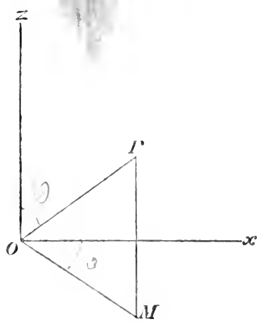
Thus, the point P may be considered as the intersection of the three planes, whose equations are

$$x = a, \quad y = b, \quad z = c.$$

The points l, O may be denoted by $(a, 0, 0)$ and $(0, 0, 0)$ and the points L and M by $(0, b, c)$ and $(a, 0, c)$.

Polar Coordinate System.

5. In the system of Polar Coordinates, a plane zOx is chosen, and in this plane a straight line Oz is drawn from a fixed point O.



The position of a point P in space is completely determined when its distance from the fixed point O is given, the angle through which OP has revolved from Oz in a plane zOP passing through Oz, and the angle through which this plane has revolved into its position from the fixed plane of reference zOx.

These coordinates are usually denoted by the symbols r, θ, and ϕ, and the point P by (r, θ, ϕ).

Thus, if the longitude of a place be l, the latitude λ, and the radius of the earth a, we may take the first meridian for the plane zOx, the axis of the earth for the line Oz, and the position of the plane will be expressed by $(a, \frac{1}{2}\pi - \lambda, l)$. The position of Greenwich, latitude λ', is given by $(a, \frac{1}{2}\pi - \lambda', 0)$.

I.

(1) Construct the positions of points which are represented by the equations

$$x^2 - y^2 = z^2,$$
$$x + y = 4a,$$
$$x - y = a.$$

(2)
$$x^2 + y^2 = 2z^2,$$
$$x + y = 2z,$$
$$xy = a^2.$$

(3) Shew that, for every point in OP, P being (a, b, c),

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}.$$

(4) Shew that, for every point in the plane $LMlm$ in the preceding figure,

$$\frac{x}{a} + \frac{y}{b} = 1.$$

(5) Draw a figure, every point of which satisfies the equations

$$x^2 + y^2 = a^2, \quad z = 0.$$

CHAPTER II.

GENERAL DESCRIPTION OF LOCI OF EQUATIONS. SURFACES. CURVES.

Locus of an Equation.

6. If an equation $\phi(x, y, z) = 0$ be given, in which the variables are the coordinates of any point, the number of solutions of this equation is generally infinite, *i.e.* the number of points whose coordinates satisfy the equation is infinite; we shall proceed to shew what is the general nature of the distribution of these points.

We shall prove, in the first place, that no algebraical equation can be satisfied by every point of *any solid* figure, but, in the most general case, only by every point in some surface or surfaces.

7. If an equation involve only one of the coordinates as x, we know that such an equation, $\phi(x) = 0$, has a finite or an infinite number of roots $a, b, c, \ldots$ separated by definite intervals, and is equivalent to the equations $x = a$, $x = b$, ... each of which, as $x = a$, is satisfied by every point in a plane parallel to the plane yOz at an algebraical distance a from that plane. Hence, all the points, whose coordinates satisfy the equation $\phi(x) = 0$, lie in a series of planes parallel to yOz at algebraical distances $a, b, c, \ldots$.

If the given equation involve two only of the variables, as $\phi(y, z) = 0$, on the plane yz let the curve be constructed, every point of which satisfies this equation, and let a straight line be drawn parallel to Ox through any point in this curve, then every point in this line is such that its coordinates, as well as those of the point through which it is drawn, satisfy the given equation, and the same is true of all points in the curve, but of no other points. Hence, all the points

which satisfy the proposed equation lie in a surface generated by a straight line parallel to Ox, which passes successively through every point of the curve traced on the plane yz; such a surface is called a *cylindrical* surface, and the curve is called the *trace* on the plane yz, being one of an infinite number of curves called *guiding curves* to the cylindrical surface. The number of guiding curves is infinite, since, if any curve be traced upon the cylindrical surface so as to cross every generating line, a line, moving parallel to Ox so as to traverse every portion of such a curve traced in space, would generate the entire cylindrical surface, that curve serving to guide the direction of motion of the generating line.

8. We may notice here that if the equation $\phi(y, z) = 0$ be equivalent to a series of equations of such forms as

$$(y - b)^2 + (z - c)^2 = 0,$$

$$(my - nz)^2 + (z - c)^2 = 0,$$

the trace on yz is reduced to a series of points, and the locus of the equation $\phi(y, z) = 0$ becomes a series of straight lines parallel to Ox and passing through those points.

In such cases the locus appears to be different in character from that of the general case, since it is a series of lines instead of being a surface. But it may be seen that this is only in appearance, since each of the equations, whose locus is called a point, represents a closed curve of infinitely small dimensions, and the lines are cylinders whose breadths are infinitely small, and the locus of the equation $\phi(y, z) = 0$ is, as in the general case, a series of surfaces, and a similar interpretation may be given in every case.

9. We shall now proceed to the general case, in which all the coordinates are involved,

$$\phi(x, y, z) = 0,$$

and examine the position of all the points which satisfy the equation.

We shall first find the position of those points which are at an algebraical distance f from the plane of yz, which is the

same thing as finding those points of the locus which lie in a plane whose equation is $x = f$.

Such points are contained in the cylindrical surface whose equation is $\phi(f, y, z) = 0$. The trace of this surface on the plane $x = f$ is the line which contains all the points of the locus which lie on that plane; and, if the series of lines be supposed traced corresponding to different positions of the plane $x = f$, for values of f varying from $-\infty$ to $+\infty$, we shall evidently obtain a surface which will contain all the points which satisfy the equation

$$\phi(x, y, z) = 0.$$

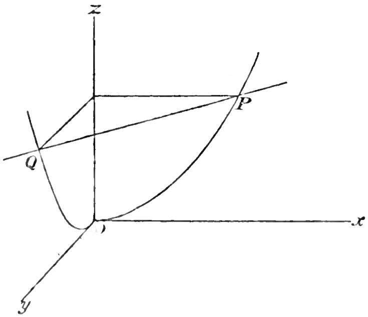
10. As an illustration of tracing surfaces, we will take the case of the surface whose equation is

$$(x + y)^2 = az.$$

If $x = 0$, $y^2 = az$, therefore the trace upon the plane of yz is a parabola whose axis is Oz and vertex O.

Similarly the trace on zx is an equal parabola having the same axis and vertex.

If $z = h$, $(x + y)^2 = ah$, the latter is the equation of two planes parallel to Oz, equally inclined to the planes yz, zx, therefore



the trace on the plane $z = h$ is two straight lines equally inclined to the planes of yz, zx.

Hence, the surface may be generated by straight lines such

as PQ, which move parallel to the plane xy, constantly passing through the traces OP, OQ on the planes zx, yz, and inclined to these planes at equal angles of 135°.

The shape is therefore a cylindrical surface.

Locus of the Polar Equation.

11. We shall examine in order the loci of equations in Polar coordinates which involve one or more of these coordinates.

(1) If the equation be $F(r)=0$, this is equivalent to a series of equations $r=a$, $r=b$, ... any one of which being satisfied the original equation is satisfied; $r=a$ is satisfied by all points at a distance a from the origin, measured in any direction; therefore the locus of $F(r)=0$ is a series of concentric spheres, whose centre is the origin.

(2) If the equation be $F(\theta)=0$, it is equivalent to $\theta=\alpha$, $\theta=\beta$, ..., any one $\theta=\alpha$ is satisfied by every point of lines through O inclined to Oz at angles equal to α; therefore the locus $F(\theta)=0$ is a series of conical surfaces, whose common axis is Oz, common vertex O, and vertical angles 2α, 2β, ...

(3) If the equation be $F(\phi)=0$, it is equivalent to $\phi=\alpha$, $\phi=\beta$, ..., any one $\phi=\alpha$ is satisfied by every point in a plane through Oz inclined at an angle α to the fixed plane zOx; therefore the locus of $F(\phi)=0$ is a series of planes through Oz inclined to zOx at angles α, β, ...

(4) If the equation involve only r and θ, as $F(r, \theta)=0$, since for all values of ϕ the same relation exists between r and θ, the locus of the equation is the surface generated by the revolution of a curve traced on a plane passing through Oz, as this plane revolves about Oz as an axis.

(5) If the equation involve only θ and ϕ, as $F(\theta, \phi)=0$, for every value of ϕ, there is a series of values of θ, corresponding to which if straight lines be drawn through O, every point in these lines will be such that its coordinates will satisfy the equation, and as ϕ changes, or the plane through Oz revolves, these lines assume new positions relative to Oz, and generate, during the revolution of the plane, conical surfaces, a conical

surface being defined to be a surface generated by a straight line moving in any manner with the restriction that it passes through a fixed point.

(6) If the only coordinate involved be r, ϕ, as in $F(r, \phi)=0$, for each position of the plane through Oz inclined at any angle ϕ to the plane zOx, there is a series of values of r which are constant for all values of θ, *i.e.* there is a series of concentric circles in the plane, the coordinates of each point in which satisfy the equation.

The locus of the equation is therefore a surface generated by circles having their centres in O, and varying in magnitude as their planes revolve about the line Oz through which they pass.

(7) If the equations involve all the coordinates, as $F(r, \theta, \phi)=0$, let any value, as β, be given to ϕ, then corresponding to this value there is a plane through Oz, and if the locus of $F(r, \theta, \beta)=0$ be traced on this plane, and such curves be drawn upon all planes corresponding to values of ϕ from $-\infty$ to $+\infty$, the surface which contains all these curves will be the locus of the equation.

Curves.

12. Curves in space are called generally *curves of double curvature*, because generally they do not lie entirely in one plane, so that if we take three points very near to one another, these three points lie in one plane, but not generally in one straight line, while a fourth point will lie generally on one side or the other of this plane, the bend first in one plane and then in another giving rise to the term *double curvature*.

Equations of Curves.

13. Through every curve there can be drawn an infinite number of surfaces, the intersections of any two of which will include every point of the curve. At the same time we must observe that two surfaces, each of which contains a given curve, may not be sufficient to determine the position of the

curve definitely, because they may intersect in other points which are not connected with the given curve.

Thus, if we take the case of a circle, it is true that it lies entirely in the intersection of a certain sphere and cylinder, but the sphere and cylinder are not sufficient to determine the circle without ambiguity, because they also intersect in another circle. It is possible, however, in this case to find two surfaces which do define the circle completely, as, for example, a plane and either a sphere or cylinder.

14. If $\phi(x, y, z) = 0$ and $\psi(x, y, z) = 0$ be equations of two surfaces, these surfaces, by their intersection, determine a certain curve, and if another equation $\chi(x, y, z) = 0$ be derived from these equations by any algebraical process, this third equation will be satisfied by every point in the curve determined by the intersection of the first two surfaces, and we may employ this equation and either of the first two to obtain properties of the curve, although the new equations which we employ may represent surfaces which intersect in other points than those of the curve originally proposed.

For example, the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

$$\text{and } x^2 + y^2 + z^2 = b^2$$

represent two surfaces, the first of which is called an ellipsoid, the second is a sphere, now, dividing the second equation by b^2, and, subtracting the first from it, we obtain the equation

$$\left(\frac{1}{b^2} - \frac{1}{a^2}\right) x^2 = \left(\frac{1}{c^2} - \frac{1}{b^2}\right) z^2,$$

which, if $a > b > c$, represents two planes and shews that the curve of intersection is composed of two circles, which are the intersection of the sphere and the two planes.

15. It is often convenient in practice to consider a curve as the intersection of two cylindrical surfaces, whose generating lines are parallel to two of the coordinate axes. In this way

of considering curves, the equations of the surfaces are of the form

$$\phi(x, z) = 0,$$
$$\psi(y, z) = 0.$$

As a simple example of this method the straight line joining the points n, N in the figure on page 2 is determined by the two plane surfaces whose equations are

$$\frac{x}{a} + \frac{z}{c} = 1,$$

$$\text{and } \frac{y}{b} + \frac{z}{c} = 1.$$

II.

Trace the surfaces represented by the equations

(1) $$x^2 + y^2 = ax.$$

(2) $$z^2 = ax + by.$$

(3) $$x^2 + y^2 + z^2 = 2ax + 2by + 2cz.$$

(4) $$x^2 + y^2 = az.$$

(5) $$xz^2 = c^2y.$$

(6) $$xy = az.$$

(7) $$c^2y^2 = (c - z)^2 (a^2 - x^2).$$

(8) $$(x + y)^2 = c(z - x).$$

(9) Shew that the surfaces

$$(x + y)^2 = a(z - x),$$
$$\text{and } x + y - z = 0,$$

intersect in a parabolic cylinder.

(10) Describe the three surfaces

$$r = a \sin\theta,$$
$$r = a \cos\phi,$$
$$4\theta = 2\pi + \pi \sin 4\phi.$$

CHAPTER III.

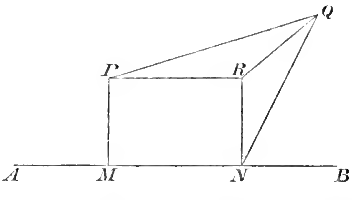
PROJECTIONS OF LINES AND AREAS. DIRECTION-COSINES AND DIRECTION-RATIOS.

16. **Def.** The *geometrical projection* of a straight line of limited length upon any other straight line given in position is the distance intercepted between the feet of the perpendiculars let fall from the extremities of the limited line upon the straight line on which it is to be projected.

17. *The geometrical projection of a straight line of limited length on a given straight line is equal to the given length multiplied by the cosine of the acute angle contained between the lines.*

Let PQ be the line of limited length, AB the indefinite line upon which it is to be projected.

Let QRN be a plane through Q perpendicular to AB meeting it in N, PR parallel to AB meeting QRN in R.



Therefore PR being parallel to AB is perpendicular to the plane QRN, and therefore to RN and QR, and QN is perpendicular to AB; hence, if PM be drawn perpendicular to AB, MN is the projection of PQ, and QPR is the acute angle contained between PQ and AB, and since $PRNM$ is a rectangle,

$$MN = PR = PQ \cos QPR.$$

If PQ produced intersects AB, the proposition is obviously true.

18. Def. The *algebraical projection* of a line PQ upon an indefinite line AB given in position is the projection estimated in a given direction, as AB.

If α be the angle through which PQ may be supposed to have revolved from PR, drawn in the positive direction AB, the algebraical projection of $PQ = PQ \cos\alpha$.

If N lies in the opposite direction with reference to M, α is obtuse, and $PQ \cos\alpha$ is negative.

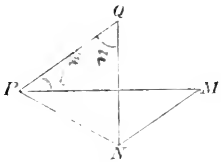
The algebraical projection of a limited straight line upon a line given in position measures the distance traversed in the direction of the latter line in passing from one extremity of the former to the other.

This consideration shews that, if all the sides of a closed polygon taken in order be projected on any straight line given in position, the sum of the algebraical projections of these sides is zero; since, in passing round the perimeter of the polygon from any point, the whole distance advanced in any direction is zero.

Hence, the algebraical projection of any side AB of a closed polygon is the sum of the algebraical projections of the remaining sides commencing from A and terminating in B.

Note. In future, when the term projection is used, the algebraical projection is to be understood.

19. Let PQ be any line, PM, MN, NQ three straight lines drawn in any given directions so as to terminate in Q, and l, m, n the cosines of the angles which PQ makes with these directions.



Then PQ will be the sum of the projections of PM, MN, and NQ on PQ; therefore $PQ = l\,.\,PM + m\,.\,MN + n\,.\,NQ$.

Direction-Cosines.

20. The direction of a straight line in space is determined when the angles which it makes with the coordinate axes are known.

DEF. If the coordinate axes be perpendicular, the cosines of the inclinations to the three axes are called *direction-cosines*.

21. *To find the relation between the direction-cosines of a straight line.*

If l, m, n be the direction-cosines of PQ, and PM, MN, NQ be parallel to the coordinate axes,

$$PM = PQ.l, \quad MN = PQ.m, \quad NQ = PQ.n.$$

Join PN, then, since QN is perpendicular to NM, MP, and therefore to the plane PMN, PNQ is a right angle;

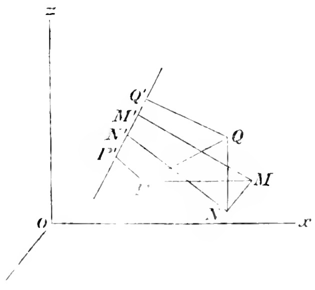
$$\text{hence } PQ^2 = PN^2 + NQ^2 = PM^2 + MN^2 + NQ^2;$$

$$\therefore \ 1 = l^2 + m^2 + n^2,$$

which is the relation required. Hence the three angles of inclination cannot all be assumed arbitrarily.

22. *To find the angle between two straight lines in terms of their direction-cosines.*

Let PQ, $P'Q'$ be two straight lines whose direction-cosines are (l, m, n) and (l', m', n') respectively.



Let PM, MN, NQ be drawn parallel to the axes, connecting

any two points P, Q, and PP', QQ' perpendicular to $P'Q'$, and let θ be the angle between PQ and $P'Q'$.

Then $P'Q'$, the projection of PQ on $P'Q'$, will be equal to the sum of the projections of PM, MN, NQ on $P'Q'$, namely, $P'M'$, $M'N'$, $N'Q'$;

$$\text{therefore } PQ\cos\theta = PM.l' + MN.m' + NQ.n',$$

$$\text{and since } PM = PQ.l,\ MN = PQ.m,\ NQ = PQ.n,$$

$$\therefore\ \cos\theta = ll' + mm' + nn';$$

$$\text{hence, } \sin^2\theta = (l^2 + m^2 + n^2)(l'^2 + m'^2 + n'^2) - (ll' + mm' + nn')^2$$

$$= (mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2.$$

When $\theta = \frac{1}{2}\pi$, $ll' + mm' + nn' = 0$, the condition that the two straight lines may be at right angles.

23. *To find the direction-cosines of a straight line perpendicular to two straight lines whose direction-cosines are given.*

Let l, m, n, and l', m', n' be the given direction-cosines, and λ, μ, ν the required cosines of the perpendicular.

Then from the condition of perpendicularity,

$$l\lambda + m\mu + n\nu = 0,$$

$$\text{and } l'\lambda + m'\mu + n'\nu = 0,$$

$$\text{whence } (ln' - l'n)\lambda + (mn' - m'n)\mu = 0,$$

$$\text{and } \frac{\lambda}{mn' - m'n} = \frac{\mu}{nl' - n'l} = \frac{\nu}{lm' - l'm}, \text{ by symmetry,}$$

$$= \pm\frac{1}{\sin\theta}, \text{ (Art. 22)},$$

if θ be the angle between the lines.

24. *To find the direction-cosines of two straight lines which lie in the plane containing two straight lines, whose direction-cosines are given, and bisect the angles between them.*

Let AP, AQ be the two given lines, whose direction-cosines are l, m, n and l', m', n'.

Take $AP = AQ = r$, join PQ and bisect it in R, AR is one of the bisecting lines, let its direction-cosines be λ, μ, ν, and if 2θ be the angle between AP and AQ, $AR = r\cos\theta$.

If AP, AQ, and AR be projected upon the axis Ox, the projection of R bisects that of PQ;

$$\therefore\ 2r\cos\theta.\lambda = lr + l'r, \text{ and } \lambda = \frac{l+l'}{2\cos\theta},$$

and similarly for μ and ν.

Produce QA to q so that $Aq = r$, and bisect Pq in r, Ar is the other bisector, and since the direction-cosines of Aq are $-l'$, $-m'$, $-n'$ and $Ar = 2r\sin\theta$, if λ', μ', ν' be the direction-cosines of Ar;

$$\therefore\ 2r\sin\theta.\lambda' = lr + (-l')\,r, \text{ and } \lambda' = \frac{l-l'}{2\sin\theta},$$

and similarly for μ' and ν', θ being determined by the equation

$$\cos 2\theta = ll' + mm' + nn', \text{ (Art. 22)}.$$

25. *To find the angle between the two straight lines whose direction-cosines are given by two homogeneous equations of the first and second degrees respectively.*

Let the given equations be

$$al^2 + bm^2 + cn^2 + 2a'mn + 2b'nl + 2c'lm = 0,$$

$$\text{and } \alpha l + \beta m + \gamma n = 0.$$

That there are two lines may be seen by eliminating n from the two equations, whereby we obtain the equation giving two values of $l : m$,

$$\gamma^2(al^2 + bm^2 + 2c'lm) - 2\gamma(\alpha l + \beta m)(b'l + a'm) + c(\alpha l + \beta m)^2 = 0,$$

$$\text{or } vl^2 + 2w'lm + um^2 = 0,$$

$$\text{where } v = a\gamma^2 - 2b'\gamma\alpha + c\alpha^2,$$

$$w' = c'\gamma^2 - (a'\alpha + b'\beta)\gamma + c\alpha\beta,$$

$$u = c\beta^2 - 2a'\beta\gamma + b\gamma^2.$$

Now, let l_1, m_1, n_1, and l_2, m_2, n_2 be the direction-cosines of the two straight lines, then, $l_1 : m_1$ and $l_2 : m_2$ being roots of the equation,

$$\frac{l_1l_2}{u} = \frac{m_1m_2}{v} = \frac{l_1m_2 + l_2m_1}{-2w'} = \left\{\frac{(l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2}{4(w'^2 - uv)}\right\}^{\frac{1}{2}}.$$

Now, it can be shewn, by collecting the coefficients of the different powers of γ, that

$$w'^2 - uv = \gamma^2 (A\alpha^2 + B\beta^2 + C\gamma^2 + 2A'\beta\gamma + 2B'\gamma\alpha + 2C'\alpha\beta),$$

where $A = a'^2 - bc$ and $A' = aa' - b'c'$,

and similar expressions.

We have, therefore, from symmetry,

$$\frac{l_1 l_2}{u} = \frac{m_1 m_2}{v} = \frac{n_1 n_2}{w} = \frac{m_1 n_2 - m_2 n_1}{2\alpha P}$$

$$= \frac{n_1 l_2 - n_2 l_1}{2\beta P} = \frac{l_1 m_2 - l_2 m_1}{2\gamma P},$$

where P^2 is written for the symmetrical expression

$$A\alpha^2 + \ldots + 2A'\beta\gamma.$$

Therefore, if ϕ be the angle between the lines,

$$\frac{\cos\phi}{u+v+w} = \frac{\sin\phi}{2P(\alpha^2+\beta^2+\gamma^2)^{\frac{1}{2}}}.$$

COR. The conditions that two such equations may represent two perpendicular or two parallel directions are

$u + v + w = 0$, and $P = 0$, respectively.

The condition of perpendicularity may be written

$$(a+b+c)(\alpha^2+\beta^2+\gamma^2) - f(\alpha, \beta, \gamma) = 0,$$

if $f(l, m, n) = 0$ be the equation of the second degree. The condition of parallelism may be expressed by the determinant

$$\begin{vmatrix} a, & c', & b', & \alpha \\ c', & b, & a', & \beta \\ b', & a', & c, & \gamma \\ \alpha, & \beta, & \gamma, & 0 \end{vmatrix} = 0.$$

Direction-ratios.

26. DEF. If the coordinate axes be not perpendicular to each other, the direction of a line PQ is fully determined, when the ratios of PM, MN, NQ to PQ are given, PM, MN, NQ being parallel to the axes. These ratios are called *direction-ratios*.

27. *To find the relation between the direction-ratios of a straight line.*

In the figure on page 14, let the angles yOz, zOx, xOy be λ, μ, ν, and let α, β, γ be the angles between PQ and the axes, l, m, n the direction-ratios of PQ.

Projecting the line PQ and the bent line $PMNQ$ terminated in the same points on Ox,

$$PQ\cos\alpha = PM + MN\cos\nu + NQ\cos\mu,$$

$$\left.\begin{aligned} \therefore\quad \cos\alpha &= l + m\cos\nu + n\cos\mu;\\ \text{similarly}\quad \cos\beta &= l\cos\nu + m + n\cos\lambda,\\ \text{and}\quad \cos\gamma &= l\cos\mu + m\cos\lambda + n.\end{aligned}\right\}$$

Also, projecting $PMNQ$ on PQ,

$$PM\cos\alpha + MN\cos\beta + NQ\cos\gamma = PQ,$$

$$\therefore\; l\cos\alpha + m\cos\beta + n\cos\gamma = 1,$$

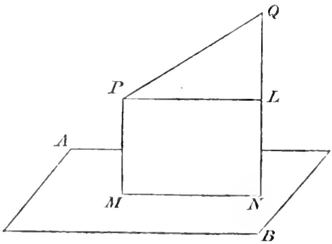
$$\therefore\; 1 = l^2 + m^2 + n^2 + 2mn\cos\lambda + 2nl\cos\mu + 2lm\cos\nu,$$

which is the relation required.

Projection of a Line on a Plane.

28. DEF. The orthogonal projection of a line of limited length on a plane is the line intercepted between the perpendiculars drawn from the extremities of the limited line upon the plane.

29. *The orthogonal projection of a line upon a plane is the length of the line multiplied by the cosine of the angle of inclination of the line to the plane.*



Let PQ be the given line, AB the plane, PM, QN perpendiculars upon the plane.

Since PM, QN are perpendicular to the plane AB, PM is parallel to QN, and the plane $MPQN$ is perpendicular to the plane AB; join MN, and draw PL parallel to MN;

$$\therefore \angle PLQ = \angle MNQ = \text{a right angle};$$

$$\therefore MN = PL = PQ \cos QPL,$$

and MN is the projection of PQ on AB,

$\angle QPL =$ the inclination of PQ to the plane,

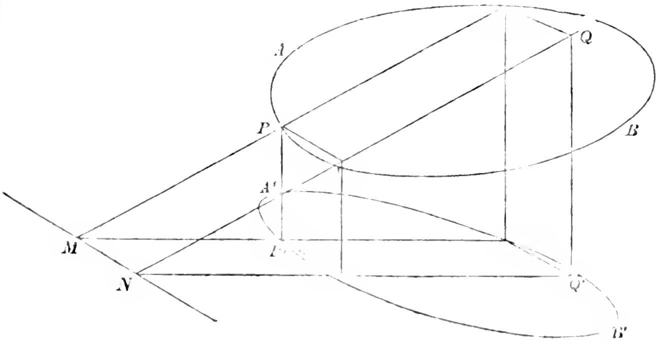
whence the proposition.

Projection of a Plane Area upon a Plane.

30. Def. The orthogonal projection of a closed plane area upon a fixed plane, is the area included within the line which is the locus of the feet of perpendiculars drawn from every point in the boundary of the plane area.

If a series of planes be taken forming a closed polyhedron, the algebraical projections of the faces upon any plane are their areas multiplied by the cosines of the angles which their normals, *drawn outwards*, make with the normal to the plane.

31. *The orthogonal projection of any plane area on a given plane is the area multiplied by the cosine of the inclination of the plane of the area to the given plane.*



Let APB be any closed curve described upon a given plane, and $A'P'B'$ the *orthogonal projection* upon any other fixed plane, which is the locus of the feet of the perpendiculars drawn to the second plane from every point of the curve APB.

The areas APB, $A'P'B'$ may have inscribed in them any number of parallelograms, such as PQ, $P'Q'$, whose sides are in planes PMP', QNQ' drawn perpendicular to the line of intersection of the given planes, and parallel to that line, and these parallelograms are in the ratio of 1 : cosine of the inclination of the planes; therefore the sums of the parallelograms are in the same ratio.

Hence, proceeding to the limit when the breadths of these parallelograms are indefinitely diminished, the area of the projection of $APB =$ area of $APB \times$ cosine of the inclination of the planes.

32. *If the faces of any closed polyhedron be projected on any plane, the sum of the algebraical projections of the faces on any fixed plane will be zero.*

One side of the fixed plane being selected as that to which the normal is drawn, the angle between this normal and the normal, drawn outwards, at any point of the closed polyhedron, is quite definite; and the projection of any face will be positive or negative according as this angle is acute or obtuse. Now any straight line whatever (produced indefinitely both ways) will meet the polyhedron in 0, 2, 4, ... or some *even* number of points, since passing from outside to inside, or from inside to outside, necessitates crossing a face once. Draw a straight line parallel to the normal to the plane of projection meeting the polyhedron in points P_1, P_2, P_3, ..., P_{2n}, and round it an indefinitely small cylinder whose transverse section is α, then the projections of the sections of this cylinder made by the faces of the polyhedron which it meets will be alternately $+\alpha$ and $-\alpha$, and since the number of them is even, their sum will always be zero. This being true for every straight line perpendicular to the plane of projection, will be true for the total projection of the polyhedron; and will also be true when the number of faces is indefinitely increased, and the areas of some,

or all of them, diminished indefinitely; that is, the sum of the algebraical projections of all the elements of a closed surface on any fixed plane is zero.*

33. *To find the area of any plane surface in terms of the areas of the projections upon any rectangular coordinate planes.*

Let l, m, n be the direction-cosines of a normal to the plane on which the given area A lies, A_x, A_y, A_z the areas of the projections upon the coordinate planes of yz, zx, xy.

Then, since l is the cosine of the angle between Ox and the normal to the plane, which is the same as the angle between the plane of A and the plane of yz, $A_x = Al$,

$$\text{and similarly, } A_y = Am, \text{ and } A_z = An;$$
$$\therefore A^2 = A^2 (l^2 + m^2 + n^2) = A_x^{\ 2} + A_y^{\ 2} + A_z^{\ 2}.$$

34. *To find the plane upon which the sum of the projections of any number of given plane areas is a maximum.*

Let A, A', A''... be any number of plane areas, l, m, n, l', m', n'... the direction-cosines of the normals to their planes, λ, μ, ν those of the normal to a plane upon which they are projected; and let A_x, A_y, A_z and A'_x, A'_y, A'_z... be the areas of the projections of the given areas upon the coordinate planes.

Then since $l\lambda + m\mu + n\nu$ is the cosine of the angle between the plane of A, and the plane upon which it is projected, the projection of A is

$$A (l\lambda + m\mu + n\nu) = A_x\lambda + A_y\mu + A_z\nu;$$

therefore the sum of the projections of all the areas upon the plane (λ, μ, ν) is $\lambda\Sigma(A_x) + \mu\Sigma(A_y) + \nu\Sigma(A_z)$ which is to be a maximum by the variation of λ, μ, ν, subject to the condition $\lambda^2 + \mu^2 + \nu^2 = 1$;

$$\therefore \Sigma(A_x)\, d\lambda + \Sigma(A_y)\, d\mu + \Sigma(A_z)\, d\nu = 0,$$
$$\text{and } \lambda d\lambda + \mu d\mu + \nu d\nu = 0,$$

must be true for an infinite number of values of $d\lambda : d\mu : d\nu$;

$$\therefore \frac{\lambda}{\Sigma(A_x)} = \frac{\mu}{\Sigma(A_y)} = \frac{\nu}{\Sigma(A_z)} = \frac{1}{\sqrt{[\{\Sigma(A_x)\}^2 + \{\Sigma(A_y)\}^2 + \{\Sigma(A_z)\}^2]}},$$

* See Thomson and Tait's Elements of Natural Philosophy, Arts. 446–450.

which determine the direction of the plane of projection, in order that the sum of the projections of the areas may be a maximum.

III.

(1) Two straight lines are drawn in the planes of xy and yz, making angles α, γ with the axes of x, z respectively; the direction-cosines of the straight line perpendicular to the two are proportional to $\tan\alpha$, -1, $\tan\gamma$.

(2) If two straight lines be inclined at an angle of 60°, and their direction-cosines be l, m, n, l', m', n', there will be a straight line whose direction-cosines are $l-l'$, $m-m'$, and $n-n'$, and this straight line will be inclined at angles of 60° and 120° to the former straight lines.

(3) If the angles which a straight line through the origin forms with the coordinate planes be in arithmetical progression, whose difference is 45°, the line must lie in one of the coordinate planes.

If it form angles α, 2α, 3α with the coordinate axes, it must lie in one of the coordinate planes.

(4) The angle between two faces of a regular tetrahedron is $\sec^{-1}3$.

(5) Find the angle between the two straight lines, whose direction-cosines are given by $l^2+m^2=n^2$ and $l+m+n=0$.

(6) Shew, by projecting upon the base, that the area of the surface of a right cone is πal, a being the radius of the base, and l the length of a slant side.

(7) Shew *a priori* that the rational equation connecting the direction-cosines of a straight line can only involve even powers of those quantities.

(8) Three circles whose areas are in the ratio 3 : 4 : 5 lie in three perpendicular planes, shew that the plane on which the sum of the projection is greatest is inclined at an angle 45° to the plane of one of the circles.

(9) If a plane mirror be equally inclined to each of the three coordinate planes, and λ, μ, ν be the direction-cosines of a ray incident on it, shew that those of the reflected ray will be

$$\tfrac{1}{3}(2\mu+2\nu-\lambda),\ \ \tfrac{1}{3}(2\nu+2\lambda-\mu),\ \text{ and }\ \tfrac{1}{3}(2\lambda+2\mu-\nu).$$

(10) If $\delta\theta$ be the small angle between two lines, whose direction-cosines are respectively l, m, n and $l+\delta l$, $m+\delta m$, $n+\delta n$, prove that

$$\overline{\delta\theta}|^2=\overline{\delta l}|^2+\overline{\delta m}|^2+\overline{\delta n}|^2.$$

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(11) Determine the plane and the area of the maximum projection of the hexagon formed by the six edges of a cube that do not meet a given diagonal.

(12) The sum of the three acute angles which a straight line forms with three rectangular coordinate axes is less than 120°.

(13) The sum of the acute angles which any straight line makes with rectangular coordinate axes can never be less than $\frac{3}{2}\sec^{-1}(-3)$.

(14) The direction-cosines of a straight line perpendicular to the two whose direction-cosines are proportional to l, m, n and $m+n, n+l, l+m$, are proportional to $m-n, n-l, l-m$.

(15) The straight lines whose direction-cosines are given by the equations

$$al + bm + cn = 0,$$

$$\alpha l^2 + \beta m^2 + \gamma n^2 = 0,$$

will be perpendicular, if $a^2(\beta+\gamma) + b^2(\gamma+\alpha) + c^2(\alpha+\beta) = 0$,

and parallel, if $\dfrac{a^2}{\alpha} + \dfrac{b^2}{\beta} + \dfrac{c^2}{\gamma} = 0.$

(16) The straight lines whose direction-consines are given by the equations

$$al + bm + cn = 0,$$

$$\frac{\alpha}{l} + \frac{\beta}{m} + \frac{\gamma}{n} = 0,$$

will be perpendicular, if $\dfrac{\alpha}{a} + \dfrac{\beta}{b} + \dfrac{\gamma}{c} = 0$,

and parallel, if $\sqrt{(a\alpha)} \pm \sqrt{(b\beta)} \pm \sqrt{(c\gamma)} = 0.$

(17) The direction-cosines of a line making equal angles with three straight lines whose direction-cosines are

$$(l, m, n),\ (l', m', n'),\ (l'', m'', n''),$$

are proportional to

$$m(n'-n'') + m'(n''-n) + m''(n-n'),$$

$$n(l'-l'') + n'(l''-l) + n''(l-l'),$$

$$l(m'-m'') + l'(m''-m) + l''(m-m').$$

If the given lines be mutually at right angles, the direction-cosines will be

$$\frac{l+l'+l''}{\sqrt{3}},\quad \frac{m+m'+m''}{\sqrt{3}},\quad \frac{n+n'+n''}{\sqrt{3}}.$$

(18) If the direction-cosines of two straight lines be given by the equations

$$amn + bnl + clm = 0,\quad \alpha l + \beta m + \gamma n = 0,$$

prove that the tangent of the angle between the lines will be

$$\frac{\{(\alpha^2+\beta^2+\gamma^2)(a^2\alpha^2 + \ldots - 2bc\beta\gamma - \ldots)\}^{\frac{1}{2}}}{a\beta\gamma + b\gamma\alpha + c\alpha\beta}.$$

(19) Find the direction-cosines of the two straight lines which are equally inclined to the axis of z, and are perpendicular to each other and to the line which makes equal angles with the coordinate axes.

(20) If A, B, C, D be four points in a plane, A', B', C', D' their projections on any other plane, the volumes of the tetrahedrons $ABCD'$, $A'B'C'D$ will be equal.

(21) If l, m, n be the cosines of the angles which a straight line makes with three oblique coordinate axes, and λ, μ, ν be the angles between the axes,

$$\begin{aligned} &l^2 \sin^2\lambda + m^2 \sin^2\mu + n^2 \sin^2\nu + 2mn\,(\cos\mu \cos\nu - \cos\lambda) \\ &\quad + 2nl\,(\cos\nu \cos\lambda - \cos\mu) + 2lm\,(\cos\lambda \cos\mu - \cos\nu) \\ &\quad = 1 - \cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda \cos\mu \cos\nu. \end{aligned}$$

(22) If A, B, C, D be the areas of the faces of a tetrahedron; a, b, c, α, β, γ the cosines of the dihedral angles (BC), (CA), (AB), (DA), (DB), (DC), respectively; then will

$$\frac{A^2}{1 - a^2 - b^2 - c^2 - 2abc} = \frac{B^2}{1 - a^2 - \beta^2 - c^2 - 2a\beta c} = \frac{C^2}{1 - a^2 - b^2 - \gamma^2 - 2ab\gamma}$$

$$= \frac{D^2}{1 - a^2 - b^2 - c^2 - 2abc}.$$

CHAPTER IV.

DIVISION OF LINES IN A GIVEN RATIO. DISTANCES OF POINTS. EQUATIONS OF A STRAIGHT LINE.

35. *To find the coordinates of a point which divides the straight line joining two given points in a given ratio.*

Let the given points be $P(x, y, z)$, and $P'(x', y', z')$, and let Q divide PP' in a given ratio, so that $PQ : QP' :: \lambda' : \lambda$. If M, N, M' be the feet of the ordinates of P, Q, P' parallel to Oz, and mQm' parallel to MNM' meet MP in m, and $M'P'$ in m',

$Pm : m'P' :: PQ : QP' :: \lambda' : \lambda$; $\therefore$ if ξ, η, ζ be coordinates of Q,

$$\lambda(\zeta - z) = \lambda'(z' - \zeta); \quad \therefore \ \zeta = \frac{\lambda z + \lambda' z'}{\lambda + \lambda'},$$

and similarly for ξ and η.

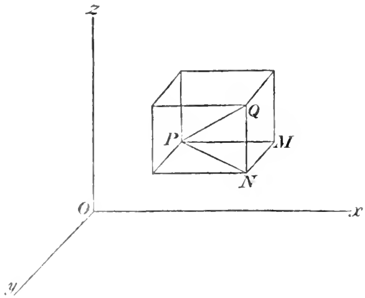
When Q lies in PP' produced in the direction of P', PQ and QP' being measured in opposite directions are affected with opposite signs and λ is negative. In like manner, when Q is in PP' produced in the direction of P, λ' is negative. In all cases due regard being paid to the signs of λ and λ' have

$$\frac{PQ}{\lambda'} = \frac{QP'}{\lambda} = \frac{PP'}{\lambda + \lambda'}.$$

Distance between two points.

36. *To find the distance between two points whose coordinates are given, referred to rectangular axes.*

Let (x, y, z), (x', y', z') be two points P, Q whose coordinates are given referred to a rectangular system; and let a parallelepiped be constructed whose diagonal is PQ, and whose edges PM, MN, NQ are parallel to the coordinates axes Ox, Oy, Oz; and join PN.



Then, since QN is perpendicular to the plane PMN, and therefore to PN,

$$PQ^2 = PN^2 + QN^2,$$

$$\text{but } PN^2 = PM^2 + MN^2;$$

$$\therefore PQ^2 = PM^2 + MN^2 + NQ^2.$$

PM is the difference of the algebraical distances of Q and P from the plane yOz, and similarly for MN, NQ:

$$\therefore PQ^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2.$$

If α, β, γ be the inclinations of PQ to the axes of coordinates,

$$x' - x = PQ \cos\alpha,$$

$$y' - y = PQ \cos\beta,$$

$$z' - z = PQ \cos\gamma,$$

$$\therefore 1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma.$$

The double sign, which appears in the value of PQ, may be interpreted in a manner similar to that adopted in the case of the radius vector in polar coordinates in Plane Geometry.

If the angles α, β, γ define the direction of measurement of the distance PQ of Q from P, the opposite direction is defined by $\pi + \alpha$, $\pi + \beta$, $\pi + \gamma$, and therefore these angles with an algebraical distance $-PQ$ equally determine the position of the point Q with reference to P.

The distance of the point (x', y', z') from the origin is

$$\sqrt{(x'^2 + y'^2 + z'^2)}.$$

37. *To find the distance between two points referred to oblique axes.*

Let λ, μ, ν be the angles between the axes; and (x, y, z), (x', y', z') two points P and Q.

Let a parallelepiped be constructed whose diagonal is PQ, and whose edges PM, MN, NQ are parallel to Ox, Oy, Oz.

Now, the projections on PM of the line PQ, and of the bent line $PMNQ$ terminated in the same points, are equal.

Therefore if α, β, γ be the angles which PQ makes with the axes,

$$\left.\begin{aligned} PQ\cos\alpha &= PM + MN\cos\nu + NQ\cos\mu, \\ \text{similarly}\quad PQ\cos\beta &= MN + NQ\cos\lambda + PM\cos\nu, \\ \text{and}\quad PQ\cos\gamma &= NQ + PM\cos\mu + MN\cos\lambda, \end{aligned}\right\} \qquad (1).$$

Also PQ is the projection of $PMNQ$ on PQ;

$$\therefore\ PQ = PM\cos\alpha + MN\cos\beta + NQ\cos\gamma \qquad (2).$$

Therefore multiplying the equations (1) by PM, MN, NQ we have by (2),

$$\begin{aligned} PQ^2 = PM^2 + MN^2 + NQ^2 &+ 2MN.NQ\cos\lambda \\ &+ 2NQ.PM\cos\mu + 2PM.MN\cos\nu, \end{aligned}$$

and PM is the difference of the algebraical distances of Q and P from yOz, and therefore $= x' - x$, and similarly $MN = y' - y$, and $NQ = z' - z$;

$$\begin{aligned} \therefore\ PQ^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2 &+ 2(y'-y)(z'-z)\cos\lambda \\ &+ 2(z'-z)(x'-x)\cos\mu + 2(x'-x)(y'-y)\cos\nu, \end{aligned}$$

whence PQ is determined as required.

38. If l, m, n be the direction-ratios of PQ,

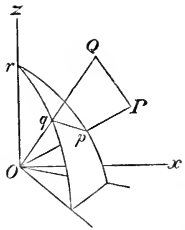
$$PM = l.PQ,\quad MN = m.PQ,\quad NQ = n.PQ;$$

$$\therefore\ 1 = l^2 + m^2 + n^2 + 2mn\cos\lambda + 2nl\cos\mu + 2lm\cos\nu,$$

which is the equation connecting the direction-ratios of any line referred to oblique axes.

39. *To find the distance of two points whose polar coordinates are given.*

Let (r, θ, ϕ) and (r', θ', ϕ') be the given points P and Q.



Join OP, OQ, QP, and let a spherical surface, whose centre is O and radius unity, intersect OP, OQ, and OZ in p, q, and r.

Then, $rp = \theta$, $rq = \theta'$, and $\angle qrp = \phi' - \phi$.

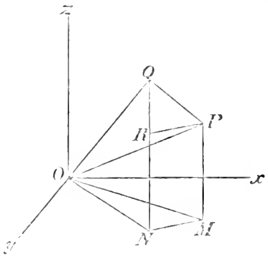
$$PQ^2 = OP^2 + OQ^2 - 2\,OP.OQ \cos pq$$
$$= r^2 + r'^2 - 2rr' \cos pq.$$

But $\cos pq = \cos pr \cos qr$

$$+ \sin pr \sin qr \cos prq$$
$$= \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi' - \phi)\};$$

$$\therefore\ PQ^2 = r^2 + r'^2 - 2rr' \{\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi' - \phi)\},$$

whence the distance PQ is determined in terms of the polar coordinates of P and Q.



40. The distance may be determined without Spherical Trigonometry as follows.

Draw PM, QN perpendicular to the plane of xy, join MN, OM and ON, and draw PR perpendicular to QN;

$$\therefore\ PQ^2 = QR^2 + PR^2$$
$$= QR^2 + MN^2,$$
$$QR = r' \cos\theta' - r\cos\theta,$$

and $MN^2 = OM^2 + ON^2 - 2\,OM.ON \cos MON$

$$= r^2 \sin^2\theta + r'^2 \sin^2\theta' - 2rr' \sin\theta \sin\theta' \cos(\phi' - \phi);$$

$$\therefore\ PQ^2 = r^2 + r'^2 - 2rr' \{\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi' - \phi)\},$$

which gives the required distance.

The Straight Line.

41. The general equations of the straight line which will be employed are of two forms: one form is symmetrical, and the equations are deduced from the consideration that the position of a straight line is completely determined, when one point in the line is given, and the direction in which the straight line

is drawn. The symmetry of this form gives great advantages, and in all questions of a general nature the general symmetrical equations will be almost exclusively employed. The other form is unsymmetrical, and the equations are deduced from the consideration that a straight line is the intersection of two planes, and is completely determined when the equations of the two planes are given. These equations in their simplest forms are the equations of planes parallel to two of the coordinate axes, and are the same as the equations of the projections of the straight line parallel to these axes upon two of the coordinate planes. It will be seen that, in cases in which the elimination of the constants is an essential part of the solution of a problem, the unsymmetrical equations may be used with advantage.

42. *To find the symmetrical equations of a straight line.*

Let A be a fixed point (a, b, c) of a straight line, P any other point (x, y, z), l, m, n the direction-cosines of AP; and let $AP = r$.

Then the projection of AP on the axis of x is $x - a$, and it is also lr, hence $\frac{x-a}{l} = r$, and, similarly, $\frac{y-b}{m} = r$, and also $\frac{z-c}{n} = r$. The equations of the straight line are therefore

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n},$$

$$\text{or } \frac{x-a}{L} = \frac{y-b}{M} = \frac{z-c}{N},$$

if L, M, N are any quantities proportional to l, m, n.

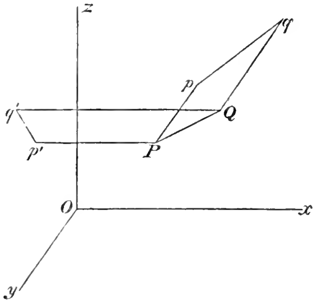
It should be carefully remembered that, when the former equations are used, each member of the equations is equal to the distance r of the current point (x, y, z) from the fixed point (a, b, c).

The equations of a straight line will be of the same form if the axes be oblique, the same interpretation being given to r, and l, m, n being the direction-ratios. The projections employed in the above proof will then be the intercepts on

the axes made by planes through A and P parallel to the coordinate planes.

43. *To find the non-symmetrical equations of a straight line.*

If a straight line PQ be projected by straight lines parallel to the axes Oy, Ox, whether rectangular or oblique, on the two coordinate planes xz, yz, each projection will be a straight line, as pq, $p'q'$, in these planes respectively.



Hence, the coordinates x, z of any point (x, y, z) in PQ being the same as those of the projection of the point in pq, satisfy an equation of the form $x = pz + h$, and the coordinates y, z similarly an equation of the form $y = qz + k$; and, consequently, the equations of the line may be written

$$x = pz + h, \quad y = qz + k.$$

44. *On the number of independent constants employed in the equation of a straight line.*

It may be noticed that the latter system of equations involves only four constants, whilst the symmetrical system involves six.

Of the three l, m, n, however, we know that they are connected by the relation $l^2 + m^2 + n^2 = 1$ (Art. 21) or an equivalent relation (Art. 27) if the axes be oblique, which renders them equivalent to only two independent constants; and, if we take L, M, N, since they are only required to be

proportional to l, m, n, one of these may be assumed arbitrarily, and they are still equivalent to two constants only.

Also, of the three a, b, c, one may be assumed at pleasure; for, since the straight line cannot be parallel to all the coordinate planes, let it not be parallel to that of yz; then at whatever distance a from yz we take a parallel plane, the straight line will meet this plane, and we may take the point where they meet for the point (a, b, c), that is, we may give to a any value we please, and the three a, b, c are consequently equivalent to two independent constants only.

45. *To find the equations of a straight line parallel to a coordinate plane.*

If a straight line be parallel to a coordinate plane, as that of yz, every point in it is at a constant distance from this plane, and we have the equation $x = h$, therefore the equations will be of the form

$$x = h, \quad y = qz + k.$$

Taking the symmetrical form, since the line will be perpendicular to the axis of x, $l = 0$, and therefore $L = 0$, and the equations of the line assume the form

$$\frac{x-a}{0} = \frac{y-b}{m} = \frac{z-c}{n} = r,$$

$$\text{or} \quad \frac{x-a}{0} = \frac{y-b}{M} = \frac{z-c}{N},$$

which form implies that $x = a$ for every point in the line at a finite distance, since the members are not infinite for such values.

46. *To find the equations of a straight line parallel to one of the coordinate axes.*

If the straight line be parallel to one of the coordinate axes, it will be parallel to the two coordinate planes passing through that axis, and consequently any point in it will be at an invariable distance from each of these planes. Hence, if a straight line be parallel to the axis of z, the distances of any

point in it from the planes yz, xz will be constant, a fact expressed by the equations

$$x = h, \quad y = k,$$

which will, therefore, be the equations of the line.

As before, the symmetrical form is

$$\frac{x-a}{0} = \frac{y-b}{0} = \frac{z-c}{N}.$$

47. *To find the angle between two straight lines whose equations are given.*

If the equations of a straight line be given in the form

$$\frac{x-a}{L} = \frac{y-b}{M} = \frac{z-c}{N},$$

then, if l, m, n be its direction-cosines,

$$\frac{l}{L} = \frac{m}{M} = \frac{n}{N} = \frac{\pm\sqrt{(l^2+m^2+n^2)}}{\sqrt{(L^2+M^2+N^2)}} = \frac{\pm 1}{\sqrt{(L^2+M^2+N^2)}},$$

or the direction-cosines will be

$$\frac{\pm L}{\sqrt{(L^2+M^2+N^2)}}, \quad \frac{\pm M}{\sqrt{(L^2+M^2+N^2)}}, \quad \frac{\pm N}{\sqrt{(L^2+M^2+N^2)}} \qquad (1).$$

If the equations be given in the form

$$x = pz + h, \quad y = qz + k,$$

since these may be written

$$\frac{x-h}{p} = \frac{y-k}{q} = \frac{z}{1},$$

the direction-cosines of the line will be

$$\frac{\pm p}{\sqrt{(p^2+q^2+1)}}, \quad \frac{\pm q}{\sqrt{(p^2+q^2+1)}}, \quad \frac{\pm 1}{\sqrt{(p^2+q^2+1)}} \qquad (2).$$

In (1) and (2) the ambiguities have the same sign.

Hence, if the equations of two straight lines be

$$\frac{x-a}{L} = \frac{y-b}{M} = \frac{z-c}{N},$$

$$\frac{x-a'}{L'} = \frac{y-b'}{M'} = \frac{z-c'}{N'},$$

the angle between them will be

$$\cos^{-1}\frac{LL'+MM'+NN'}{\sqrt{(L^2+M^2+N^2)}\sqrt{(L'^2+M'^2+N'^2)}}, \text{ (Art. 22)}.$$

And, if the equations be

$$x=pz+h,\quad y=qz+k,$$
$$x=p'z+h',\quad y=q'z+k',$$

the angle between them will be

$$\cos^{-1}\frac{pp'+qq'+1}{\sqrt{(p^2+q^2+1)}\sqrt{(p'^2+q'^2+1)}}.$$

48. *To find the conditions that two straight lines whose equations are given may be parallel.*

If the two straight lines

$$\frac{x-a}{L}=\frac{y-b}{M}=\frac{z-c}{N},$$

$$\frac{x-a'}{L'}=\frac{y-b'}{M'}=\frac{z-c'}{N'},$$

be parallel, they will have the same direction-cosines, and, since L, M, N and also L', M', N' are respectively proportional to these direction-cosines,

$$\frac{L}{L'}=\frac{M}{M'}=\frac{N}{N'} \tag{1}$$

will be the conditions of parallelism.

These conditions may be derived from the general value of the cosine of the angle between them, which will then be unity.

$$\text{For } 1=\frac{LL'+MM'+NN'}{\sqrt{(L^2+M^2+N^2)}\sqrt{(L'^2+M'^2+N'^2)}},$$

or $(L^2+M^2+N^2)(L'^2+M'^2+N'^2)-(LL'+MM'+NN')^2=0$,

or $(LM'-L'M)^2+(MN'-M'N)^2+(NL'-N'L)^2=0$,

which is equivalent to the conditions (1).

Similarly, if the straight lines

$$x=pz+h,\quad y=qz+k,$$
$$x=p'z+h',\quad y=q'z+k'$$

be parallel,

$$p=p', \quad q=q',$$

which results follow from the consideration that, if the straight lines be parallel, their projections will also be parallel.

49. *To find the condition that two straight lines, whose equations are given, may be perpendicular.*

If the straight lines be perpendicular, the cosine of the angle between them will vanish, and the condition that this may be the case is

$$LL'+MM'+NN'=0, \quad \text{or } pp'+qq'+1=0,$$

according to the systems of equations given.

50. *To find the condition that two straight lines, whose equations are given, may intersect.*

Let the equations of the two straight lines be

$$\frac{x-a}{L}=\frac{y-b}{M}=\frac{z-c}{N}, \tag{1}$$

$$\frac{x-a'}{L'}=\frac{y-b'}{M'}=\frac{z-c'}{N'}, \tag{2}$$

and let each member of (1) be equal to R, and of (2) equal to R'.

Then, if the lines intersect, equations (1) and (2) must be simultaneously satisfied by the coordinates of the point in which they intersect.

Hence,

$$a'-a+L'R'-LR=0,$$
$$b'-b+M'R'-MR=0,$$
$$c'-c+N'R'-NR=0,$$

and eliminating R and R', we obtain the required condition

$$\begin{vmatrix} a'-a, & L', & L \\ b'-b, & M', & M \\ c'-c, & N', & N \end{vmatrix}=0,$$

With the equations

$$x=pz+h, \quad y=qz+k,$$
$$x=p'z+h', \quad y=q'z+k',$$

the condition is immediately found to be

$$\frac{h'-h}{p'-p}=\frac{k'-k}{q'-q}$$

by eliminating x, y, and z.

Straight line under given conditions.

51. *To find the equations of a straight line passing through a given point.*

If (a, b, c) be the given point, we have already seen that the symmetrical form

$$\frac{x-a}{L}=\frac{y-b}{M}=\frac{z-c}{N}$$

will represent a straight line passing through that point. The unsymmetrical form is

$$x-a=p(z-c),\quad y-b=q(z-c).$$

52. *To find the equations of a straight line passing through a given point and parallel to a given straight line.*

The equations of a straight line passing through a given point (a, b, c) are

$$\frac{x-a}{L}=\frac{y-b}{M}=\frac{z-c}{N},$$

and if this be parallel to a straight line whose direction-cosines are l, m, n,

$$\frac{L}{l}=\frac{M}{m}=\frac{N}{n},\quad \text{(Art. 18)}$$

therefore the required equations will be

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}.$$

53. *To find the equations of a straight line passing through a given point, and perpendicular to and intersecting a given straight line.*

Let (a, b, c) be the given point, and the equations of the given straight line be

$$\frac{x-a'}{l}=\frac{y-b'}{m}=\frac{z-c'}{n}.$$

Then $\dfrac{x-a}{L}=\dfrac{y-b}{M}=\dfrac{z-c}{N}$

will be the required equations of the straight line, where the ratios $L:M:N$ are to be determined from the equations

$$Ll+Mm+Nn=0, \text{ (Art. 49)}$$

$$\begin{vmatrix} a'-a, & l, & L \\ b'-b, & m, & M \\ c'-c, & n, & N \end{vmatrix}=0. \text{ (Art. 50)}$$

54. *To find the equations of a straight line passing through a point and intersecting two given straight lines.*

Let (a, b, c) be the given point, and let the equations of the two given straight lines be

$$\frac{x-a'}{L'}=\frac{y-b'}{M'}=\frac{z-c'}{N'}, \text{ and } \frac{x-a''}{L''}=\frac{y-b''}{M''}=\frac{z-c''}{N''};$$

and let the equations of the straight line satisfying the required conditions being

$$\frac{x-a}{L}=\frac{y-b}{M}=\frac{z-c}{N}.$$

By the conditions of intersection given in Art. 50 L, M, N satisfy the equations

$$LP'+MQ'+NR'=0,$$

$$LP''+MQ''+NR''=0,$$

where P', Q', R', &c., are the first minors of the two corresponding determinants, whence the equations of the straight line become

$$\frac{x-a}{Q'R''-Q''R'}=\frac{y-b}{R'P''-R''P'}=\frac{z-c}{P'Q''-P''Q'}.$$

55. *To find the equations of a straight line passing through a given point, parallel to a given plane, and intersecting a given straight line.*

Let (a, b, c) be the given point, l, m, n the direction-cosines of a normal to the plane, which will therefore be perpendicular

to the straight line whose equations are required, and let the equations of the given straight line be

$$\frac{x-a'}{l'}=\frac{y-b'}{m'}=\frac{z-c'}{n'}.$$

The required equations will then be

$$\frac{x-a}{L}=\frac{y-b}{M}=\frac{z-c}{N},$$

where $L : M : N$ are determined by the equations

$$Ll+Mm+Nn=0,$$

$$\text{and}\quad \begin{vmatrix} a'-a, & l' & L \\ b'-b, & m', & M \\ c'-c, & n', & N \end{vmatrix}=0.$$

56. *To find the distance from a given point to a given straight line.*

Let A be the given point (x', y', z'),

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n},$$

the equations of the given straight line, B being the point (a, b, c); AP the perpendicular from A on the straight line; then the projections of BA on the axes of x, y, z are respectively $x'-a$, $y'-b$, $z'-c$; and the projections of these on the given line are $l(x'-a)$, $m(y'-b)$, $n(z'-c)$, but the sum of these projections is the projection of BA on the straight line, or

$$BP=l(x'-a)+m(y'-b)+n(z'-c);$$

hence, $AP^2=BA^2-BP^2$

$$=(x'-a)^2+(y'-b)^2+(z'-c)^2-\{l(x'-a)+m(y'-b)+n(z'-c)\}^2,$$

giving the required distance, which may be written

$$\sqrt{[\{n(y'-b)-m(z'-c)\}^2+\{l(z'-c)-n(x'-a)\}^2+\{m(x'-a)-l(y'-b)\}^2]}.$$

If the equations of the line be

$$x=pz+h,\quad y=qz+k,$$

which may be written

$$\frac{x-h}{p}=\frac{y-k}{q}=\frac{z}{1},$$

the distance will be

$$\sqrt{\left[(x'-h)^2+(y'-k)^2+z'^2-\frac{\{p(x'-h)+q(y'-k)+z'\}^2}{p^2+q^2+1}\right]}.$$

57. *To find the equation of a circular cylinder, the equations of whose axis and the radius of a circular section of which are given.*

The circular cylinder being the locus of a point whose distance from the axis is constant and equal to the given radius r, if the equations of the axis be

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n},$$

the equation of the surface will be, by the preceding article,

$$(x-a)^2+(y-b)^2+(z-c)^2-\{l(x-a)+m(y-b)+n(z-c)\}^2=r^2.$$

58. *To find the equation of a circular cone, whose vertex, vertical angle, and the equations of whose axis are given.*

If V be the vertex, P any point of the cone, PQ perpendicular on the axis, and 2α the vertical angle,

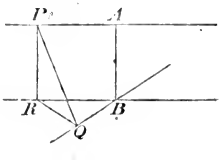
$$VQ^2=VP^2\cos^2\alpha;$$

therefore, if (a, b, c) be the vertex, the equations of the axis as before, the equation of the cone will be

$$\{l(x-a)+m(y-b)+n(z-c)\}^2=\cos^2\alpha\,\{(x-a)^2+(y-b)^2+(z-c)^2\}.$$

59. *To show that the shortest distance between two straight lines which do not intersect is perpendicular to both.*

Let AP, BQ be the two straight lines, and let a plane be drawn through BQ parallel to AP, and BR be the orthogonal projection of AP upon this plane, B being the projection of A; therefore AB will be perpendicular to both straight lines, for it meets two parallel lines AP, BR, to one of which, BR, it is perpendicular, and it is also perpendicular to BQ, since it is drawn perpendicular to the plane QBR.



Let P, Q be any points in AP, BQ, join PQ, draw PR perpendicular to BR, and join QR; then PQ is greater than PR, being opposite to the greater angle, and $PR = AB$; therefore AB is less than PQ, or the distance which is perpendicular to both straight lines is less than any other distance.

60. *To find the shortest distance between two straight lines whose equations are given.*

Let the equations of the two straight lines be

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}, \text{ and } \frac{x-a'}{l'} = \frac{y-b'}{m'} = \frac{z-c'}{n'},$$

and let λ, μ, ν be the direction-cosines of the straight line perpendicular to each, then (Art. 23)

$$\frac{\lambda}{mn'-m'n} = \frac{\mu}{nl'-n'l} = \frac{\nu}{lm'-l'm} = \frac{1}{\sin\theta},$$

θ being the angle between the lines.

Now, if we suppose P, Q, in the last figure, to be the points (a, b, c), (a', b', c'), the projection of PQ on AB, which is AB itself, will be $\lambda(a-a') + \mu(b-b') + \nu(c-c')$, hence

$$AB = \frac{(a-a')(mn'-m'n) + (b-b')(nl'-n'l) + (c-c')(lm'-l'm)}{\{(mn'-m'n)^2 + (nl'-n'l)^2 + (lm'-l'm)^2\}^{\frac{1}{2}}}.$$

61. *To find equations of the line on which lies the shortest distance between two straight lines whose equations are given.*

Taking the equations of the last article, if (ξ, η, ζ) be any point of the line considered, the equation of the line will be

$$\frac{x-\xi}{mn'-m'n} = \frac{y-\eta}{nl'-n'l} = \frac{z-\zeta}{lm'-l'm}.$$

Hence, by Art. 50, since it meets each of the two given lines, we have

$$\begin{vmatrix} \xi-a, & \eta-b, & \zeta-c \\ l, & m, & n \\ mn'-m'n, & nl'-n'l, & lm'-l'm \end{vmatrix} = 0$$

and

$$\begin{vmatrix} \xi-a', & \eta-b', & \zeta-c' \\ l', & m', & n' \\ mn'-m'n, & nl'-n'l, & lm'-l'm \end{vmatrix} = 0,$$

and, since (ξ, η, ζ) is any point on the line these are equations of the line.

If l, m, n and l', m', n' be direction-cosines, since

$$m(lm' - l'm) - n(nl' - n'l) = l(mm' + nn') - l'(m^2 + n^2)$$
$$= l(ll' + mm' + nn') - l',$$

these equations may be written

$$(l\cos\theta - l')(x-a) + (m\cos\theta - m')(y-b) + (n\cos\theta - n')(z-c) = 0,$$
$$(l'\cos\theta - l)(x-a') + (m'\cos\theta - m)(y-b') + (n'\cos\theta - n)(z-c') = 0,$$

where θ is the angle between the given lines.

62. A very simple form, in which the equations of two straight lines can be presented, will be obtained by taking the middle point of the shortest distance between them for the origin, the line in which it lies for one of the axes, suppose that of z, and the two planes equally inclined to the two straight lines for those of zx, zy.

If 2α be the angle between the two straight lines, $2c$ the shortest distance between them, their equations will then become

$$y = x\tan\alpha,\ z = c, \text{ and } y = -x\tan\alpha,\ z = -c.$$

IV.

(1) The straight line given by the equations

$$x + 2y + 3z = 0,\ 3x + 2y + z = 0,$$

makes equal angles with the axes of x and z, and an angle $\sin^{-1}\dfrac{1}{\sqrt{3}}$ with the axis of y.

(2) Prove that the equations $\dfrac{x^2-1}{x-1} = \dfrac{y^2-1}{y-1} = \dfrac{z^2-1}{z-1}$ represent seven straight lines which all pass through the same point.

(3) Find the direction-cosines of the straight line determined by the equations

$$lx + my + nz = mx + ny + lz = nx + ly + mz.$$

(4) The angle between the two straight lines given by the equations

$$x = y \text{ and } xy + yz + zx = 0,$$

is $\sec^{-1} 3$.

(5) Find the equations of the straight line passing through the points (b, c, a) (c, a, b), and shew that it is perpendicular to the line passing through the origin and through the middle point of the line joining the two points, and also to each of the straight lines whose equations are

$$x = y = z, \quad \frac{x}{a} = \frac{y}{b} = \frac{z}{c}.$$

(6) Find the shortest distance between the axis of z and the straight line $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z}{n}$, and find its equations.

(7) Find the distance between the two parallel straight lines

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}, \quad \frac{x-a'}{l} = \frac{y-b'}{m} = \frac{z-c'}{n},$$

and the equation of the containing plane.

(8) Find the shortest distance between a side of a cube and a diagonal which does not meet it.

(9) Prove that the equations of any straight line intersecting the two straight lines $y = mx$, $z = c$; $y = -mx$, $z = -c$; may be written in the form

$$\frac{x - \frac{\lambda}{m}\cos\theta}{\sin\theta} = \frac{y - m\lambda\sin\theta}{\cos\theta} = \frac{\lambda z}{c}.$$

(10) The equations of two straight lines are

$$\frac{x}{\sin a} = \frac{y}{\cos a} = \frac{z-c}{0},$$

$$\frac{x}{\sin a} = \frac{y}{-\cos a} = \frac{z-c}{0};$$

shew that the distance between two points on these straight lines whose distances from the axis of z are a, b respectively is

$$\sqrt{(4c^2 + a^2 + b^2 \mp 2ab\cos 2a)}.$$

(11) Interpret the equation

$$(x^2 + y^2 + z^2)(l^2 + m^2 + n^2) = (lx + my + nz)^2,$$

and give a geometrical illustration.

(12) The locus of the middle points of all straight lines terminated by two fixed straight lines is a plane bisecting the shortest distance between the fixed straight lines.

(13) Find the equations of the straight line which passes through the origin and intersects at right angles the straight line whose equations are

$$(m+n)\,x + (n+l)\,y + (l+m)\,z = a,$$

$$(m-n)\,x + (n-l)\,y + (l-m)\,z = a;$$

and obtain the coordinates of the point of intersection.

(14) The equations $\dfrac{x^3+1}{x+1} = \dfrac{y^3+1}{y+1} = \dfrac{z^3+1}{z+1}$ denote thirteen straight lines. Shew that four are equally inclined to each other, and construct for the rest.

(15) The straight lines determined by the equations

$$lx + my + nz = 0,$$

$$l\,(b-c)\,yz + m\,(c-a)\,zx + n\,(a-b)\,xy = 0,$$

are at right angles to each other.

(16) Shew that the equations

$$\frac{a+mz-ny}{l} = \frac{b+nx-lz}{m} = \frac{c+ly-mx}{n}$$

are reducible to $\dfrac{x+nb-mc}{l} = \dfrac{y+lc-na}{m} = \dfrac{z+ma-lb}{n}$,

l, m, and n being direction-cosines.

(17) The equations of a straight line are given in the form

$$\frac{a-ny+mz}{\lambda} = \frac{b-lz+nx}{\mu} = \frac{c-mx+ly}{\nu},$$

obtain them in the form

$$\frac{x - \dfrac{\mu c - \nu b}{l\lambda + m\mu + n\nu}}{l} = \frac{y - \dfrac{\nu a - \lambda c}{l\lambda + m\mu + n\nu}}{m} = \frac{z - \dfrac{\lambda b - \mu a}{l\lambda + m\mu + n\nu}}{n}.$$

(18) When a ray of light is reflected from a plane mirror, the shortest distance between the incident ray and any straight line on the mirror is equal to that between the reflected ray and the same straight line.

(19) Find the coordinates of the centre of perpendiculars of the triangle which the coordinate planes cut off from the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

(20) ABC, $A'B'C'$ are two straight lines, BB' the shortest distance between them, C, C' any two points on the two lines, such that CA' is perpendicular to $A'B'C'$ and $C'A$ to ABC; prove that

$$AB\,.\,BC = A'B'\,.\,B'C'.$$

(21) The cosine of the angle between the two straight lines whose equations are $lx + my + nz = 0$, $ax^2 + by^2 + cz^2 = 0$,

$$\text{is} \quad \frac{l^2(b+c) + m^2(c+a) + n^2(a+b)}{\sqrt{\{l^4(b-c)^2 + \ldots + 2m^2n^2(a-b)(a-c) + \ldots\}}}.$$

(22) The locus of the middle points of all straight lines of constant length terminated by two fixed straight lines, is an ellipse whose centre bisects the shortest distance between the fixed lines, and whose axes are equally inclined to them.

(23) If the axes of coordinates be inclined at angles α, β, γ, shew that the equations of the four straight lines, each point of which is equidistant from the three coordinate planes, will be

$$\frac{x^2}{\sin^2\alpha} = \frac{y^2}{\sin^2\beta} = \frac{z^2}{\sin^2\gamma}.$$

(24) If a system of straight lines be represented by

$$y = \lambda x + \mu, \quad z = \lambda' x + \mu',$$

where λ, μ, λ', μ' are given functions of a single parameter, what is the condition that any two consecutive lines of the system intersect?

CHAPTER V.

GENERAL EQUATION OF THE FIRST DEGREE. EQUATION OF A PLANE.

63. *The locus of the general equation of the first degree is a plane.*

The general equation of the first degree is

$$Ax + By + Cz + D = 0. \tag{1}$$

Let (a, b, c), (a', b', c') be two points in the locus, the equations of the straight line joining these points are

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = r, \tag{2}$$

$$\text{where } \frac{a'-a}{l} = \frac{b'-b}{m} = \frac{c'-c}{n}, \tag{3}$$

and, since (a, b, c) is in the locus of the equation (1),

$$Aa + Bb + Cc + D = 0, \tag{4}$$

$$\text{similarly, } Aa' + Bb' + Cc' + D = 0,$$

$$\therefore A(a'-a) + B(b'-b) + C(c'-c) = 0,$$

whence, the conditions (3) give

$$Al + Bm + Cn = 0. \tag{5}$$

Now the straight line (2) meets the locus of (1) in all points for which the equation in r

$$A(a + lr) + B(b + mr) + C(c + nr) + D = 0,$$

is satisfied, *i.e.* for all values of r, by (4) and (5); therefore, every point in the straight line lies in the locus, and this is true wherever the two points (a, b, c), (a', b', c') are chosen. Hence, the locus is a plane.

64. The student will readily deduce the following special positions of the plane.

(1) If $D=0$, the plane passes through the origin.

(2) If $A=0$, the plane is parallel to the axis of x.

(3) If A and $B=0$, the plane is parallel to the plane of xy.

(4) If A, B and $D=0$, the plane is that of xy.

(5) If A, B and $C=0$, while D remains finite, the plane is at an infinite distance. For, the point in which the axis of x meets the plane is given by the equations

$$y=0,\quad z=0,\quad Ax+D=0;$$

hence, the distance from the origin being $-\dfrac{D}{A}$, if A be indefinitely diminished, while D is finite, the plane cuts the axis of x at an infinite distance from the origin, and the same being true for each axis, it follows that the plane is at an infinite distance from the origin.

65. It is important to observe that the existence of three arbitrary constants in the general equation of the first degree, viz. the three ratios $A:B:C:D$, shews that a plane may be made to satisfy three conditions, provided each condition is one which gives only one relation between A, B, C, D. Thus, passing through a given point at a finite or infinite distance is such a condition, but being parallel to a given plane is equivalent to two such conditions.

Equation of a Plane.

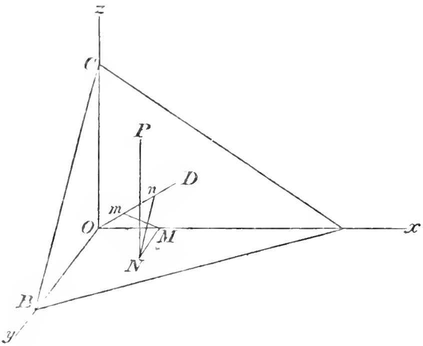
66. *To find the equation of a plane in the form*

$$lx+my+nz=p,$$

in which p is the perpendicular from the origin upon the plane, and l, m, n its direction-cosines.

A plane may be considered as the locus of a straight line which passes through a given point, and is perpendicular to a given straight line.

Let $OD=p$ be the perpendicular from the origin upon a



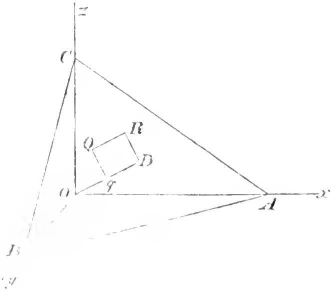
plane, l, m, n its direction-cosines, (x, y, z) any point P in the plane, then, by the definition, PD is perpendicular to OD, and OD is the sum of projections of the coordinates of P on OD;

$$\therefore \; lx + my + nz = p,$$

which is the equation of the plane in the form required, in which, if the axes be rectangular, $l^2 + m^2 + n^2 = 1$.

67. *Interpretation of the expression* $p - lx - my - nz$.

The equation $p - lx - my - nz = 0$ represents a plane, in



which p is the perpendicular from the origin, and l, m, n are its direction-cosines.

Let ABC be this plane, and suppose OD, QR to be drawn perpendicular to it, in the direction defined by (l, m, n), from the origin, and from the point $Q(x, y, z)$, and join RD, which will be perpendicular to OD. Let $QR = q$, and project x, y, z and q on OD, then $p = lx + my + nz + q$;

$$\therefore\ q = p - lx - my - nz.$$

Hence, the expression $p - lx - my - nz$ represents the perpendicular drawn from (x, y, z) upon the plane

$$p - lx - my - nz = 0,$$

estimated positive in the direction defined by the cosines l, m, and n.

68. *To find the angle between two planes whose equations are given.*

Let $Lx + My + Nz = D$, and $L'x + M'y + N'z = D'$,

be the given equations; then (L, M, N) and (L', M', N') are proportional respectively to the direction-cosines of the normals; but the angle between two planes is equal to the angle between their normals, hence the angle between the planes is

$$\cos^{-1} \frac{LL' + MM' + NN'}{\sqrt{(L^2 + M^2 + N^2)}\sqrt{(L'^2 + M'^2 + N'^2)}}.$$

The conditions of parallelism and perpendicularity are therefore respectively

$$\frac{L}{L'} = \frac{M}{M'} = \frac{N}{N'},$$

$$\text{and } LL' + MM' + NN' = 0.$$

The student may also deduce the conditions of parallelism from the consideration that parallel planes intersect in a straight line at infinity, or directly from the parallelism of the normals.

69. *To find the angle between a straight line and plane whose equations are given.*

$$\text{Let } \frac{x-a}{L} = \frac{y-b}{M} = \frac{z-c}{N}, \tag{1}$$

$$L'x + M'y + N'z = D, \tag{2}$$

be the given equations. The angle between a straight line and a plane is the complement of the angle between the straight line and the normal to the plane; hence the required angle is

$$\sin^{-1}\frac{LL'+MM'+NN'}{\sqrt{(L^2+M^2+N^2)}\sqrt{(L'^2+M'^2+N'^2)}}.$$

70. *To determine the perpendicular from a point* (f, g, h) *upon a plane whose equation is* $Ax+By+Cz+D=0$.

If we compare the equation

$$Ax+By+Cz+D=0$$

with the equation of the plane in the form

$$lx+my+nz-p=0;$$

$$\text{then } \frac{l}{A}=\frac{m}{B}=\frac{n}{C}=\frac{p}{-D}=\pm\frac{1}{\sqrt{(A^2+B^2+C^2)}}.$$

where, if the ambiguous sign be so taken that p shall be an absolute length, l, m, n will be completely determined.

The perpendicular from (f, g, h) upon the plane, estimated positive when drawn in the direction defined by these cosines,

$$= p - lf - mg - nh$$

$$= \frac{Af+Bg+Ch+D}{\mp\sqrt{(A^2+B^2+C^2)}},$$

that sign being chosen which is the same as that of D.

71. *To find the distance from a given point to a given plane, measured in any given direction.*

Let the equation of the plane be

$$Ax+By+Cz+D=0,$$

and let (f, g, h) be the given point, (l, m, n) the given direction, l, m, n being direction-cosines for rectangular axes, and direction-ratios for oblique.

The equations of a line drawn through (f, g, h) in the given direction are

$$\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n}=r,$$

and where this straight line meets the plane,

$$A(f+lr)+B(g+mr)+C(h+nr)+D=0;$$

$$\therefore \text{ the required distance is } -\frac{Af+Bg+Ch+D}{Al+Bm+Cn}.$$

Hence, if the given direction be perpendicular to the plane, and the axes be rectangular,

$$\frac{l}{A}=\frac{m}{B}=\frac{n}{C}=\frac{Al+Bm+Cn}{A^2+B^2+C^2}=\frac{1}{\pm\sqrt{(A^2+B^2+C^2)}},$$

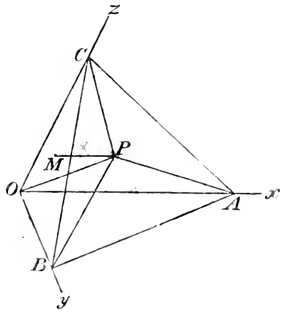
and the perpendicular distance will be $\dfrac{Af+Bg+Ch+D}{\mp\sqrt{(A^2+B^2+C^2)}}$,

the sign being chosen so that $\dfrac{D}{\pm\sqrt{(A^2+B^2+C^2)}}$ is positive.

72. *To find the equation of a plane in the form*

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

Let $OA=a$, $OB=b$, $OC=c$ be the intercepts on the axes of x, y, z by the plane ABC, and let PA, PB, PC, PO be drawn from the point $P(x, y, z)$ in the plane.



Draw PM parallel to xO, meeting yOz in M. Since the pyramids $POBC$, $AOBC$ are on the same base,

$$\text{vol } POBC : \text{vol } OABC :: PM : AO :: x : a;$$

$$\therefore \frac{x}{a}=\frac{\text{vol } POBC}{\text{vol } OABC}.$$

$$\text{Similarly, } \frac{y}{b} = \frac{\text{vol } POCA}{\text{vol } OABC},$$

$$\text{and } \frac{z}{c} = \frac{\text{vol } POAB}{\text{vol } OABC},$$

$$\text{and vol } POBC + \text{vol } POCA + \text{vol } POAB = \text{vol } OABC;$$

$$\therefore \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

which is the equation required.

The student is recommended to investigate this equation by the employment of a figure in which P lies in another compartment, as $x'y'z$, of the coordinate planes, taking care to interpret the geometrical into algebraical distances.

73. If q be the perpendicular from a point Q (x, y, z) on the plane ABC estimated in the direction of p, the perpendicular from O on the plane,

$$\frac{q}{p} = \frac{\text{vol } QABC}{\text{vol } OABC}$$

$$= 1 - \frac{x}{a} - \frac{y}{b} - \frac{z}{c}.$$

74. The equation of Art. 72 may be obtained from the general equation of the first degree.

For let a, b, c be the intercepts on the axes of x, y, z,

$$Ax + By + Cz + D = 0, \text{ the equation of the plane.}$$

Since $(a, 0, 0)$ is a point in the plane,

$$-D = Aa, \text{ and, similarly, } = Bb = Cc.$$

Hence, the equation of the plane is $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1.$

75. *To find the equation of the plane in the form*

$$z = px + qy + c.$$

Consider the plane as a surface generated by a straight line which moves subject to the conditions that it always intersects one given straight line and is parallel to another.

Let the equations of the line which it intersects be

$$z = px + c, \quad y = 0; \tag{1}$$

and those of the line to which it is parallel

$$z = qy, \quad x = 0,$$

the equations of the moving line will therefore be of the form

$$z = qy + \beta, \quad x = \alpha, \tag{2}$$

and, since the two lines, whose equations are (1) and (2), intersect,

$$\beta = p\alpha + c;$$

therefore, for every point in the plane, $z - qy = px + c$; that is, the equation of the plane is

$$z = px + qy + c.$$

In this form of the equation, c is the intercept on the axis of z cut off by the plane, p, q are the tangents of the angles made respectively with the axes of x and y by the traces on the planes of zx, yz, if the coordinates be rectangular; and the ratios of the sines of the angles made with the axes in those planes, if the coordinates be oblique.

76. *To find the polar equation of a plane.*

Let c, α, β be the polar coordinates of the foot of the perpendicular from the origin on the plane; r, θ, ϕ those of any point in the plane, then if ψ be the angle between the lines joining these points to the origin,

$$c = r\cos\psi,$$

and $\cos\psi = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos(\phi - \beta)$, (Art. 39)

whence $\dfrac{c}{r} = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos(\phi - \beta)$,

the most convenient form of the equation of a plane when referred to polar coordinates.

Planes under Particular Conditions.

77. *Equation of a plane passing through a given point.*

Let a, b, c be the coordinates of the given point, and the equation of the plane $lx + my + nz = p$, then since (a, b, c) is a point in this plane $la + mb + nc = p$, or, eliminating p,

$$l(x - a) + m(y - b) + n(z - c) = 0$$

is the general equation of a plane passing through the point (a, b, c).

78. *Equation of a plane passing through a point determined by the intersection of three given planes.*

If the point be given by the equations of three planes,

$$u = 0, \quad v = 0, \quad w = 0,$$

passing through it and not intersecting in one straight line, then $\lambda u + \mu v + \nu w = 0$ will be the general equation of a plane passing through that point, for it is satisfied by the values of x, y, z, which are given by the equations

$$u = 0, \quad v = 0, \quad w = 0,$$

taken simultaneously, and therefore passes through the intersection of these planes, which is the given point; and since this equation is of the first degree, and involves two arbitrary constants, namely, the ratios $\lambda : \mu : \nu$, it is the general equation of a plane passing through the given point.

If the three planes, $u = 0$, $v = 0$, $w = 0$, intersect in a straight line, then these equations, and therefore the equation $\lambda u + \mu v + \nu w = 0$, will be simultaneously satisfied for all points lying in that straight line. Hence, $\lambda u + \mu v + \nu w = 0$ cannot be the *general* equation of a plane passing through a given point. The position of a point is not, in this case, completely determined by the given equations, but only the fact that it lies on a certain straight line.

79. *Equation of a plane passing through two given points.*

Let (a, b, c), (a', b', c') be the given points; the equation of a plane passing through (a, b, c) is

$$l(x - a) + m(y - b) + n(z - c) = 0.$$

If this plane also pass through (a', b', c'), we shall have

$$l(a' - a) + m(b' - b) + n(c' - c) = 0,$$

which is the condition to which $l : m : n$ are subject; or, the equation of the plane may be written

$$\lambda \frac{x - a}{a' - a} + \mu \frac{y - b}{b' - b} + \nu \frac{z - c}{c' - c} = 0,$$

λ, μ, ν, being subject to the condition

$$\lambda + \mu + \nu = 0.$$

It is easily seen that if the points be given by the two systems of planes,

$$u = 0, \quad v = 0, \quad w = 0,$$

$$\text{and} \quad u = a, \quad v = b, \quad w = c,$$

that the equation of the plane will be

$$\lambda u + \mu v + \nu w = 0,$$

subject to the condition

$$\lambda a + \mu b + \nu c = 0.$$

80. *Equation of a plane passing through the line of intersection of two planes.*

If $u = 0$, $v = 0$ be the equations of the two planes, the equation $\lambda u + \mu v = 0$ will represent a plane passing through their line of intersection; and since this equation involves one arbitrary constant $(\lambda : \mu)$, it will be the *general* equation of a plane passing through the straight line which is given by the two planes.

81. *To find the equations of two planes which form an harmonic system with two given planes.*

These two planes must pass through the line of intersection of the given planes, and divide the angles between them, so that the sines of the angles made by each with the given planes shall be in the same ratio.

Let $u = 0$, $v = 0$ be the equations of the given planes, and let ρ, σ be multipliers, such that ρu and σv are reduced to the form $p - lx - my - nz$; in this form they are the perpendiculars from (x, y, z) on the given planes. Hence, it is evident that $\rho u : \pm\sigma v$ are each numerically equal to the given ratio.

The forms of the equations are therefore

$$u - kv = 0 \text{ and } u + kv = 0.$$

82. *Equation of a plane passing through three given points.*

Let (a, b, c), (a', b', c'), (a'', b'', c'') be the three given points,

$$l(x - a) + m(y - b) + n(z - c) = 0,$$

the equation of a plane passing through (a, b, c).

If this plane also pass through (a', b', c') and (a'', b'', c''), we shall have

$$l(a' - a) + m(b' - b) + n(c' - c) = 0,$$
$$l(a'' - a) + m(b'' - b) + n(c'' - c) = 0,$$

and eliminating l, m, n between (1), (2), and (3), we obtain

$$\begin{aligned} &(x-a)\{b(c'-c'') + b'(c''-c) + b''(c-c')\} \\ &+ (y-b)\{c(a'-a'') + c'(a''-a) + c''(a-a')\} \\ &+ (z-c)\{a(b'-b'') + a'(b''-b) + a''(b-b')\} = 0, \quad (1) \end{aligned}$$

as the equation of the plane passing through three given points.

The coefficients of x, y, z in this equation are the projections on the coordinate planes of the triangle formed by the three given points, call these A_x, A_y, A_z; then xA_x will be equal to three times the volume of the pyramid whose base is A_x, and vertex the point (x, y, z).

Hence, equation (1) asserts that the algebraical sum of the pyramids whose bases are the projections of any triangle on the coordinate planes, and common vertex any point in the plane of the triangle, is constant for all positions of this point.

The equation here obtained becomes nugatory if

$$b(c' - c'') + b'(c'' - c) + b''(c - c') = 0,$$
$$c(a' - a'') + c'(a'' - a) + c''(a - a') = 0,$$
and $$a(b' - b'') + a'(b'' - b') + a''(b - b') = 0,$$

which are equivalent to

$$(b - b')(c'' - c') - (c - c')(b'' - b') = 0,$$
$$(c - c')(a'' - a') - (a - a')(c'' - c') = 0,$$
$$(a - a')(b'' - b') - (b - b')(a'' - a') = 0,$$

or to $$\frac{a - a'}{a' - a''} = \frac{b - b'}{b' - b''} = \frac{c - c'}{c' - c''},$$

and these are the conditions that the three given points should lie in a straight line.

83. *To find the equation of a plane passing through a given point and parallel to a given plane.*

If (a, b, c) be the given point, and l, m, n the direction-cosines

of a normal to the given plane, the equation of the proposed plane will be

$$l(x-a)+m(y-b)+n(z-c)=0.$$

84. *To find the equation of a plane which passes through two given points and is parallel to a given straight line.*

Let (a, b, c) (a', b', c') be the given points, and l, m, n the direction-cosines of the given line, the equation of the plane will be of the form

$$\lambda(x-a)+\mu(y-b)+\nu(z-c)=0,$$

$$\text{where } \lambda(a'-a)+\mu(b'-b)+\nu(c'-c)=0, \tag{1}$$

and since its normal is perpendicular to the given line

$$\lambda l+\mu m+\nu n=0, \tag{2}$$

the equation is therefore

$$\begin{vmatrix} x-a, & y-b, & z-c \\ a'-a, & b'-b, & c'-c \\ l, & m, & n \end{vmatrix}=0.$$

This equation will become identical if $\dfrac{l}{a'-a}=\dfrac{m}{b'-b}=\dfrac{n}{c'-c}$, which are the conditions that the given straight line may be parallel to the line joining the two given points. The equations (1) and (2) will in this case be coincident, or every plane passing through the two points will necessarily be parallel to the given straight line, as is otherwise evident. The required equation will then be the equation of any plane passing through the two given points.

85. *To find the equation of a plane passing through a given point and parallel to two given straight lines.*

If the direction-cosines of the two straight lines be l, m, n and l', m', n', and the coordinates of the given point a, b, c, the equation of the plane will be

$$(mn'-m'n)(x-a)+(nl'-n'l)(y-b)+(lm'-l'm)(z-c)=0. \text{ (Art. 23)}.$$

If $\dfrac{l}{l'}=\dfrac{m}{m'}=\dfrac{n}{n'}$, this equation will be satisfied for all values of x, y, z; or, if the given straight lines be parallel, there

will be an infinite number of planes satisfying the given conditions, the direction of the normal to the required plane being indeterminate.

86. *To find the equation of a plane which contains one given straight line, and is parallel to another, not in the same plane.*

Let the equations of the given straight lines be

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n},$$

$$\text{and} \quad \frac{x-a'}{l'} = \frac{y-b'}{m'} = \frac{z-c'}{n'}.$$

The plane contains the first line, and passes through the point (a, b, c), also its normal is perpendicular to each of the lines, which properties are expressed by the equations

$$\begin{aligned} &\lambda(x-a) + \mu(y-b) + \nu(z-c) = 0, \\ &\lambda l \qquad\quad + \mu m \qquad\ + \nu n \qquad\ = 0, \\ &\lambda l' \qquad\quad + \mu m' \qquad + \nu n' \qquad = 0, \end{aligned}$$

and the equation is

$$(x-a)(mn' - m'n) + (y-b)(nl' - n'l) + (z-c)(lm' - l'm) = 0.$$

The equation of the plane containing the second and parallel to the first is

$$(x-a')(mn' - m'n) + (y-b')(nl' - n'l) + (z-c')(lm' - l'm) = 0.$$

The shortest distance of the lines is the difference of the perpendiculars from the origin, estimated in one direction, giving the same result as in Art. (60).

87. *To find the equation of a plane equidistant from two given straight lines, not in the same plane.*

Let the equations of the two given straight lines be

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = r, \qquad (1)$$

$$\frac{x-a'}{l'} = \frac{y-b'}{m'} = \frac{z-c'}{n'} = r', \qquad (2)$$

(x_1, y_1, z_1) a point in (1), (x_2, y_2, z_2) a point in (2), (ξ, η, ζ) the middle point of the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2).

$$\text{Then,}\ 2\xi = x_1 + x_2 = a + a' + lr + l'r',$$

$$2\eta = y_1 + y_2 = b + b' + mr + m'r',$$

$$2\zeta = z_1 + z_2 = c + c' + nr + n'r',$$

and eliminating r and r', we obtain, for the locus of (ξ, η, ζ), the equation

$$(2\xi - a - a')(mn' - m'n) + (2\eta - b - b')(nl' - n'l)$$
$$+ (2\zeta - c - c')(lm' - l'm) = 0. \quad (3)$$

The plane represented by this equation bisects all lines joining any point of (1) to any point of (2), and therefore bisects the shortest distance between them; and since the direction-cosines of the normal to (3) are proportional to

$$mn' - m'n, \quad nl' - n'l, \quad lm' - l'm,$$

the normal is parallel to the shortest distance between the lines (Art. 60). Hence this plane bisects at right angles the shortest distance between the lines, which is clear from the geometry.

88. *To determine the conditions necessary and sufficient in order that the general homogeneous equation of the second degree may represent two real or imaginary planes.*

Let the general equation be written

$$u_2 \equiv ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 0.$$

If a be finite, the equation is equivalent to

$$(ax + c'y + b'z)^2 = (c'^2 - ab)\,y^2 + 2\,(b'c' - aa')\,yz + (b'^2 - ac)\,z^2.$$

But, if the equation represent two planes, x must be capable of being expressed as a linear function of y and z, and this can only happen when the second side of the above equation is a complete square, and therefore of the form $(py + qz)^2$, and the two planes will have equations

$$ax + c'y + b'z = \pm (py + qz);$$

every point of the line of intersection of the two planes will therefore satisfy

$$ax + c'y + b'z = 0 \text{ and } py + qz = 0.$$

By solving with respect to y and z, we obtain similar results. Hence, for every point in the line of intersection,

$$\begin{aligned} ax + c'y + b'z &= 0, \\ c'x + by + a'z &= 0, \qquad\qquad (1) \\ b'x + a'y + cz &= 0; \end{aligned}$$

therefore, by eliminating x, y, and z,

$$\begin{vmatrix} a, & c', & b' \\ c', & b, & a' \\ b', & a', & c \end{vmatrix} = 0,$$

$$\text{or } H(u_2) = abc + 2a'b'c' - aa'^2 - bb'^2 - cc'^2 = 0,$$

this is the necessary condition, which might also have been obtained from the condition for a perfect square,

$$(b'c' - aa')^2 = (c'^2 - ab)(b'^2 - ca),$$

$$\text{or } a(abc + 2a'b'c' - aa'^2 - bb'^2 - cc'^2) = 0,$$

which, since a is finite, gives the same result.

The symmetrical form of $H(u_2)$ shews that the result would have been obtained in this way whether a, b, or c were finite.

If none of them be finite, it is easily seen that a', b', or c' must be zero, and the equation will still hold.

In order that the planes may be real, it is necessary that $c'^2 - ab$, $b'^2 - ac$, and, similarly, $a'^2 - bc$ shall not be negative.

89. *When the general equation of the second degree represents two planes, to find the equations of their line of intersection in a symmetrical form.*

Any two of the equations (1) given in the last article are equations of the line of intersection. If we eliminate z from the first two of these equations, and x from the last two, we obtain the symmetrical equations of the line

$$x(b'c' - aa') = y(c'a' - bb') = z(a'b' - cc').$$

V.

(1) The equation of a plane passing through the origin, and containing the straight line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n},$$

is $\dfrac{x}{l}\left(\dfrac{b}{m} - \dfrac{c}{n}\right) + \dfrac{y}{m}\left(\dfrac{c}{n} - \dfrac{a}{l}\right) + \dfrac{z}{n}\left(\dfrac{a}{l} - \dfrac{b}{m}\right) = 0.$

Hence, find the equations of the straight line passing through the origin, and intersecting two given straight lines; and examine the case in which the straight lines are parallel.

(2) Find the equation of the plane passing through the points (a, b, c), (b, c, a), (c, a, b), and the equations of the planes, each of which passes through two of the points and is perpendicular to the former plane.

(3) The equation of a plane passing through the origin, and containing the straight line whose equations are

$x + 2y + 3z + 4 = 2x + 3y + 4z + 1 = 3x + 4y + z + 2$, is $x + y - 2z = 0$.

(4) Find the condition that four planes whose equations are given may pass through one point.

(5) The equations of three planes are

$$x + 2y - 3z = 1,$$
$$2x - 3y + 5z = 3,$$
$$7x - y - z = 2.$$

Shew that the equation of a plane, equally inclined to the three axes, and passing through their common point, is

$$x + y + z = 6.$$

(6) Shew that the locus of a point dividing the distance between any two points on the two straight lines

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}, \quad \frac{x-a'}{l'} = \frac{y-b'}{m'} = \frac{z-c'}{n'},$$

in the ratio $\lambda' : \lambda$, is the plane whose equation is

$$(mn' - m'n)\left(x - \frac{\lambda a + \lambda' a'}{\lambda + \lambda'}\right) + \&c. = 0.$$

(7) Employ Art (35) to shew that the equation $Ax + By + Cz = D$ represents a plane, according to Euclid's definition.

(8) The edges of a parallelepiped meeting in a point A are a, b, c, and a plane is drawn cutting off parts a', b', c' from these edges; prove that the plane will cut the diagonal AB in a point B', such that

$$AB = \left(\frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'}\right) AB'.$$

(9) Find the coordinates of the centre of perpendiculars of the triangle, which the coordinate planes cut from the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

(10) The equation of a plane passing through two straight lines

$$\frac{x-a}{a'} = \frac{y-b}{b'} = \frac{z-c}{c'}, \quad \frac{x-a'}{a} = \frac{y-b'}{b} = \frac{z-c'}{c},$$

$$\text{is } (bc' - b'c)\, x + (ca' - c'a)\, y + (ab' - a'b)\, z = 0.$$

Give a geometrical interpretation of the equations.

(11) Shew that the three planes

$$lx + my + nz = 0, \quad (m+n)\, x + (n+l)\, y + (l+m)\, z = 0, \quad x + y + z = 0,$$

intersect in one straight line

$$\frac{x}{m-n} = \frac{y}{n-l} = \frac{z}{l-m}.$$

(12) Shew that if the straight lines

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}, \quad \frac{x}{a\alpha} = \frac{y}{b\beta} = \frac{z}{c\gamma}, \quad \frac{x}{l} = \frac{y}{m} = \frac{z}{n},$$

lie in one plane, then $\frac{l}{\alpha}(b-c) + \frac{m}{\beta}(c-a) + \frac{n}{\gamma}(a-b) = 0.$

(13) Determine the conditions necessary in order that the planes

$$ax + c'y + b'z = 0, \quad c'x + by + a'z = 0, \quad b'x + a'y + cz = 0,$$

may have a common line of intersection, and shew that the equations of that line are

$$x\,(aa' - b'c') = y\,(bb' - c'a') = z\,(cc' - a'b').$$

Find the conditions necessary in order that the three planes may be coincident.

(14) The equation of any plane containing the straight line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \text{ is } \frac{\lambda\,(x-a)}{l} + \frac{\mu\,(y-b)}{m} + \frac{\nu\,(z-c)}{n} = 0,$$

λ, μ, ν being connected by the equations $\lambda + \mu + \nu = 0$. Hence find the equation of a plane containing one given straight line and parallel to another.

(15) A straight line is projected on a plane which always passes through a given straight line; find the locus of the projections.

(16) The equations of two lines are

$$x = y + 2a = 6\,(z-a) \text{ and } x + a = 2y = -12z.$$

Find the two planes, each containing one line and parallel to the other, and thence shew that the shortest distance of the lines is $2a$.

(17) The angular points of a tetrahedron are (1, 2, 3), (2, 3, 4), (3, 4, 1), and (4, 1, 2); find the equations of its faces, and shew that two of the dihedral angles are right angles, two supplementary and one 60°. Also, that the perpendiculars from the angular points on the opposite faces are $\frac{4}{\sqrt{(6)}}$, $\frac{4}{\sqrt{(6)}}$, $2\sqrt{(2)}$, $2\sqrt{(2)}$.

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(18) The equation of a plane passing through the origin and containing the straight line

$$\frac{\alpha + mz - ny}{l} = \frac{\beta + nx - lz}{m} = \frac{\gamma + ly - mx}{n}$$

is $(l^2 + m^2 + n^2)(\alpha x + \beta y + \gamma z) = (l\alpha + m\beta + n\gamma)(lx + my + nz)$.

(19) If A, A'; B, B'; C, C' are fixed points in any three fixed straight lines passing through a point, the intersections of the planes ABC, $A'B'C'$; $A'BC$, $AB'C'$; $AB'C$, $A'BC'$; and ABC', $A'B'C$ are four straight lines lying in a plane dividing the fixed lines harmonically.

(20) Find the equation of the plane which passes through the two parallel lines

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}; \quad \frac{x-a'}{l} = \frac{y-b'}{m} = \frac{z-c'}{n};$$

and explain the result when $\frac{a-a'}{l} = \frac{b-b'}{m} = \frac{c-c'}{n}$.

(21) The equation of the planes which pass through the straight line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n},$$

and make an angle α with the plane $l'x + m'y + n'z = 0$, is

$$\{l'(ny - mz) + m'(lz - nx) + n'(mx - ly)\}^2$$
$$= \cos^2\alpha\,(l'^2 + m'^2 + n'^2)\,\{(ny - mz)^2 + (lz - nx)^2 + (mx - ly)^2\}.$$

What limitation is there to the value of α? Shew that for the limiting values the two planes coincide.

(22) Shew that the plane containing the line $\frac{y}{b} + \frac{z}{c} = 1$, $x = 0$, and parallel to the line $\frac{x}{a} - \frac{z}{c} = 1$, $y = 0$, is $\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0$; and, if $2d$ be the shortest distance between the lines, shew that $\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$.

(23) Shew that the line represented by the equations

$$\frac{a + mz - ny}{m - n} = \frac{b + nx - lz}{n - l} = \frac{c + ly - mx}{l - m}$$

is at an infinite distance in the plane

$$x(m-n)+y(n-l)+z(l-m)=0,$$

unless $la+mb+nc=0$, when it is indeterminate.

(24) The equations

$$\frac{ax+c'y+b'z}{x}=\frac{c'x+by+a'z}{y}=\frac{b'x+a'y+cz}{z}$$

represent in general three straight lines mutually at right angles; but, if

$$a-\frac{b'c'}{a'}=b-\frac{c'a'}{b'}=c-\frac{a'b'}{c'},$$

they represent a plane and a straight line perpendicular to that plane.

(25) A straight line moves parallel to a fixed plane, and intersects two fixed straight lines not in the same plane; prove that the locus of a point, which divides the part intercepted in a constant ratio, is a straight line.

(26) l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3, are the direction-cosines of three planes at right angles to one another, and p_1, p_2, p_3 are the perpendiculars from the origin upon these planes; prove that the locus of a point equally distant from these three planes is the line

$$\frac{x-(l_1p_1+l_2p_2+l_3p_3)}{l_1+l_2+l_3}=\frac{y-(m_1p_1+m_2p_2+m_3p_3)}{m_1+m_2+m_3}=\frac{s=(n_1p_1+n_2p_2+n_3p_3)}{n_1+n_2+n_3}.$$

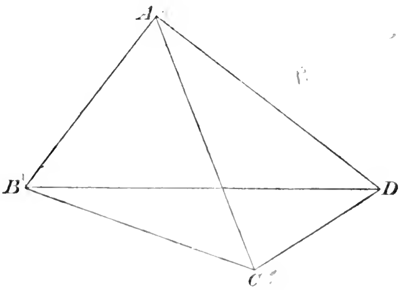
CHAPTER VI.

QUADRIPLANAR AND TETRAHEDRAL COORDINATES.

90. We now proceed to describe other systems of coordinates, which are employed in cases in which it is an object to express the relations between lines, planes, surfaces and curves by means of equations which are homogeneous in form, on account of the facilities which such forms present in the application of theorems of higher algebra.

Four-Plane or Quadriplanar System.

91. In the quadriplanar coordinate system, four planes are fixed upon as planes of reference, which form, by their intersections, a pyramid or tetrahedron $ABCD$. The position of a point is determined in this system by the algebraical distances x, y, z, w from the four planes respectively opposite to the vertices A, B, C, D, these distances being all absolute distances when the point is within the tetrahedron.



Hence, for a point in the compartment between the plane

ACD and the other three produced, y will be negative and x, z, w positive; between BAC, CAD, and DAB, produced through A, x will be positive and y, z, w all negative.

If a be positive, $x=a$ is the equation of a plane parallel to BCD, at a distance a from it, on the side towards A; $x=-a$ that of a plane on the opposite side at the same distance.

92. In this system of coordinates the following peculiarity must be observed, viz., that any three of the coordinates x, y, z, w are sufficient to determine the position of the point, since, when x, y, z are given, three planes are determined parallel to the faces opposite to A, B, C which intersect in the point, and so determine its position completely.

Hence, when x, y, z are given, w ought to be known from the geometry of the figure, and we proceed to determine the relation between the coordinates in this system.

Relation of Coordinates in the Four-Plane System.

93. Let V be the volume of the tetrahedron contained by the four fixed planes, A, B, C, D the areas of the triangular faces.

If the point P, whose coordinates are x, y, z, w, be joined by straight lines to the angular points of the tetrahedron, four pyramids will be formed, whose vertices will be at P, and whose bases will be the faces of the tetrahedron.

The algebraical sum of these four pyramids will make up the volume of the tetrahedron; therefore, remembering that the volume of a pyramid is one-third of the base × the altitude,

$$Ax+By+Cz+Dw=3V=Ap_0=Bq_0=Cr_0=Ds_0,$$

p_0, q_0, r_0, s_0 being the perpendiculars from the angular points on the opposite faces, whence, when any three of the coordinates of a point are given, the fourth may be found.

The object of the introduction of a fourth coordinate, in this system, is the same as that for which trilinear coordinates are employed in Plane Geometry, viz. to obtain equations homogeneous with reference to the coordinates, and thus to arrive at symmetrical results.

By means of the equation given above, any equation which

does not appear in a homogeneous form can be reduced to such a form immediately.

Thus the equation $x = a$ of a plane may be reduced to the homogeneous form

$$x = a\left(\frac{x}{p_0} + \frac{y}{q_0} + \frac{z}{r_0} + \frac{w}{s_0}\right).$$

Tetrahedral Coordinates.

94. It is evident that the relation between the coordinates given in the last article would be much simplified if we were to select as coordinates

$$\frac{Ax}{3V}, \frac{By}{3V}, \frac{Cz}{3V}, \frac{Dw}{3V}, \text{ or } \frac{x}{p_0}, \frac{y}{q_0}, \frac{z}{r_0}, \frac{w}{s_0},$$

to which these are equal. Such a system of coordinates is called a system of *tetrahedral coordinates*, each coordinate being the ratio of the pyramid, whose base is a face of the tetrahedron and vertex the point considered, to the volume of the fundamental tetrahedron, sign being of course always regarded.

If (x, y, z, w) represent a point in this system

$$x + y + z + w = 1,$$

and any given equation involving four-plane coordinates may be transformed into the corresponding equation in tetrahedral coordinates by writing p_0x, q_0y, r_0z, s_0w for x, y, z, w.

Since both these systems are never employed in the same discussions, it is unnecessary to adopt a different notation for the coordinates.

95. It may be shewn, as in Art. 35, that the four-plane coordinates of a point which divides the line joining two points (x, y, z, w) and (x', y', z', w') in the ratio $\mu : \lambda$ are $\dfrac{\lambda x + \mu x'}{\lambda + \mu}$, &c.; and the same result will be true, if the coordinates be tetrahedral.

96. *To find the distance of two given points in tetrahedral coordinates.*

Let (x, y, z, w) and (x', y', z', w') be two given points P, Q.

The square of the distance between them is easily seen to be of the second degree in terms of $x-x'$, $y-y'$, $z-z'$, $w-w'$.

But
$$x+y+z+w=1=x'+y'+z'+w',$$
$$\therefore\ (x-x')+(y-y')+(z-z')+(w-w')=0\,;$$
$$\therefore\ (x-x')^2=-(x-x')(y-y')-\ldots,$$
and similarly for $(y-y')^2$, &c.

Hence, the square of the distance can be expressed in terms of the six products $(x-x')(y-y')$, &c.

Let γ be the coefficient of $(x-x')(y-y')$, and let us apply the expression to find the distance AB; now the coordinates of A and B are 1, 0, 0, 0, and 0, 1, 0, 0, hence every product but one vanishes, $\therefore\ AB^2=-\gamma$, and
$$-PQ^2=AB^2(x-x')(y-y')+AC^2(x-x')(z-z')+\ldots.$$

97. Hence we may obtain the equation of a sphere, the coordinates of whose centre are f, g, h, k, and whose radius is r,
$$-r^2=a^2(y-g)(z-h)+b^2(z-h)(x-f)+c^2(x-f)(y-g)$$
$$+a'^2(x-f)(w-k)+b'^2(y-g)(w-k)+c'^2(z-h)(w-k),$$
a, b, c being the sides of ABC opposite to A, B, C; a', b', c' the edges DA, DB, DC respectively opposite to a, b, c.

The Straight Line.

98. *To find the equations of a straight line in four-plane coordinates.*

If (f, g, h, k) be a fixed point in a straight line, (x, y, z, w) any other point, R the distance between them, λ, μ, ν, ρ the cosines of the angles between the straight line and the normals to the corresponding faces of the tetrahedron, $x-f=\lambda R$, &c.

Therefore the equations of the straight line are
$$\frac{x-f}{\lambda}=\frac{y-g}{\mu}=\frac{z-h}{\nu}=\frac{w-k}{\rho}=R,$$
where, since two equations are sufficient to determine the line, there must be a relation between λ, μ, ν, ρ.

$$\text{Now}\quad \frac{x-f}{p_0}+\frac{y-g}{q_0}+\frac{z-h}{r_0}+\frac{w-k}{s_0}=0,\ \text{(Art. 93)},$$

the relation is $\dfrac{\lambda}{p_0} + \dfrac{\mu}{q_0} + \dfrac{\nu}{r_0} + \dfrac{\rho}{s_0} = 0.$

Another relation, not homogeneous, may be obtained from the value of R, in Art. 96, changed to four-plane coordinates

$$\frac{a^2}{q_0 r_0}\mu\nu + \frac{b^2}{r_0 p_0}\nu\lambda + \frac{c^2}{p_0 q_0}\lambda\mu + \frac{a'^2}{p_0 s_0}\lambda\rho + \frac{b'^2}{q_0 s_0}\mu\rho + \frac{c'^2}{r_0 s_0}\nu\rho = -1.$$

In this form of the equations, λ, μ, ν, ρ may be called the direction-cosines of the line.

In tetrahedral coordinates the corresponding equations are, for the straight line

$$\frac{x-f}{\lambda} = \frac{y-g}{\mu} = \frac{z-h}{\nu} = \frac{w-k}{\rho} = \frac{R}{\sigma},$$

with the conditions $\lambda + \mu + \nu + \rho = 0$,

and $a^2\mu\nu + b^2\nu\lambda + c^2\lambda\mu + a'^2\lambda\rho + b'^2\mu\rho + c'^2\nu\rho = -\sigma^2$,

$\dfrac{\lambda p_0}{\sigma}$, $\dfrac{\mu q_0}{\sigma}$, $\dfrac{\nu r_0}{\sigma}$, $\dfrac{\rho s_0}{\sigma}$ being the direction-cosines.

99. The general equations of the straight line may be written in tetrahedral coordinates

$$\frac{x-f}{L} = \frac{y-g}{M} = \frac{z-h}{N} = \frac{w-k}{R},$$

where L, M, N, R satisfy the equation

$$L + M + N + R = 0;$$

and it may be observed that, if two equations $\dfrac{x-f}{L} = \dfrac{y-g}{M} = \dfrac{z-h}{N}$ be given, the fourth member may be derived from it, since each $= \dfrac{x-f+y-g+z-h}{L+M+N} = \dfrac{w-k}{R}$.

If the straight line pass through one of the angular points, as A, $(1, 0, 0, 0)$

$$\frac{x-1}{L} = \frac{y}{M} = \frac{z}{N} = \frac{w}{R}.$$

If it join the middle points of AB, CD, viz. $(\frac{1}{2}, \frac{1}{2}, 0, 0)$ and $(0, 0, \frac{1}{2}, \frac{1}{2})$,

$$\frac{x-\frac{1}{2}}{-\frac{1}{2}} = \frac{y-\frac{1}{2}}{-\frac{1}{2}} = \frac{z}{\frac{1}{2}} = \frac{w}{\frac{1}{2}}; \text{ or } x = y \text{ and } z = w.$$

100. *To find the angle between two straight lines whose equations are given in tetrahedral coordinates.*

Let the equations be

$$\frac{x-f}{\lambda}=\frac{y-g}{\mu}=\frac{z-h}{\nu}=\frac{w-k}{\rho}=\frac{R}{\sigma},$$

$$\text{and}\quad \frac{x-f'}{\lambda'}=\frac{y-g'}{\mu'}=\frac{z-h'}{\nu'}=\frac{w-k'}{\rho'}=\frac{R}{\sigma'}.$$

Take two straight lines parallel to these, passing through D and meeting ABC in P, P'.

The equation of DP is

$$\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}=\frac{w-1}{\rho}=\frac{R}{\sigma},$$

and the coordinates of P are $-\frac{\lambda}{\rho}$, $-\frac{\mu}{\rho}$, $-\frac{\nu}{\rho}$, 0, and $DP=-\frac{\sigma}{\rho}$, and similarly for P' and DP';

$$\therefore\ -PP'^2=a^2\left(\frac{\mu}{\rho}-\frac{\mu'}{\rho'}\right)\left(\frac{\nu}{\rho}-\frac{\nu'}{\rho'}\right)+b^2\left(\frac{\nu}{\rho}-\frac{\nu'}{\rho'}\right)\left(\frac{\lambda}{\rho}-\frac{\lambda'}{\rho'}\right)$$

$$+c^2\left(\frac{\lambda}{\rho}-\frac{\lambda'}{\rho'}\right)\left(\frac{\mu}{\rho}-\frac{\mu'}{\rho'}\right)=-\frac{\sigma^2}{\rho^2}-\frac{\sigma'^2}{\rho'^2}+2\frac{\sigma\sigma'}{\rho\rho'}\cos PDP',$$

whence, substituting the values of σ^2 and σ'^2 (Art. 98), we obtain

$$\pm 2\sigma\sigma'\cos PDP'=a^2(\mu\nu'+\mu'\nu)+\ldots+a'^2(\lambda\rho'+\lambda'\rho)+\ldots.$$

101. As an example of the use of this formula, we will find the angle between AD and BC, whose lengths are a' and a. For AD, $y=0$ and $z=0$, and for BC, $x=0$, $w=0$, and the values of λ, μ, ..., λ', μ', ... may be taken respectively, 1, 0, 0, -1 and 0, 1, -1, 0, $\therefore\ \sigma^2=a'^2$ and $\sigma'^2=a^2$, and, if θ be the acute angle between those lines,

$$2aa'\cos\theta=(b^2+b'^2)\sim(c^2+c'^2).$$

102. The condition of perpendicularity of two straight lines given in tetrahedral coordinates

$$\frac{x-f}{L}=\frac{y-g}{M}=\frac{z-h}{N}=\frac{w-k}{R},\ \text{and}\ \frac{x-f'}{L'}=\frac{y-g'}{M'}=\ldots,$$

$$\text{is}\ a^2(MN'+M'N)+\ldots+a'^2(LR'+L'R)+\ldots=0.$$

It may be seen, as an example, that, if the lines joining the middle points of (a, a') and (b, b') be perpendicular, c and c' will be equal.

The Plane.

103. *The general equation of the first degree represents a plane.*

Let $Ax + By + Cz + Dw = 0$ be any equation of the first degree.

If arbitrary points (f, g, h, k), (f', g', h', k') be taken, which satisfy the equation, *any* point P in the line joining them may be represented by $\left(\frac{\lambda f + \mu f'}{\lambda + \mu}, \frac{\lambda g + \mu g'}{\lambda + \mu}, \ldots\right)$, and since

$$Af + Bg + Ch + Dk = 0,$$

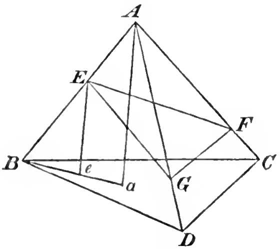
$$Af' + Bg' + Ch' + Dk' = 0;$$

$$\therefore A(\lambda f + \mu f') + B(\lambda g + \mu g') + \ldots = 0;$$

therefore the coordinates of P satisfy the equation, and the whole straight line joining any two arbitrary points lies in the locus of the equation, which is therefore a plane.

104. *Geometrical interpretation of the constants in the equation of a plane* $\lambda x + \mu y + \nu z + \rho w = 0$.

Let E be the point in which the plane EFG cuts AB, its four-plane coordinates being x', y', 0, 0; $\therefore \lambda x' + \mu y' = 0$.



Draw Ee, Aa perpendicular to BCD,

$$\therefore \frac{Ee}{Aa} = \frac{EB}{AB}, \text{ or } \frac{x'}{p_0} = \frac{EB}{AB}; \text{ and, similarly, } \frac{y'}{q_0} = \frac{EA}{BA} = \frac{AE}{AB};$$

also, if p, q be the perpendiculars from A, B upon the given plane, estimated in the same direction,

$$\frac{p}{AE} = \frac{-q}{EB}; \quad \therefore \frac{px'}{p_0} + \frac{qy'}{q_0} = 0;$$

$$\therefore \frac{\lambda}{\frac{p}{p_0}} = \frac{\mu}{\frac{q}{q_0}}, \text{ and similarly each } = \frac{\nu}{\frac{r}{r_0}} = \frac{\rho}{\frac{s}{s_0}},$$

and the equation of the plane is

$$\frac{p}{p_0} x + \frac{q}{q_0} y + \frac{r}{r_0} z + \frac{s}{s_0} w = 0.$$

In tetrahedral coordinates the equation becomes

$$px + qy + rz + sw = 0.$$

105. *To find the equation of a plane at an infinite distance.*

If the plane be at an infinite distance, $p = q = r = s$, and the equation in tetrahedral coordinates becomes

$$x + y + z + w = 0.$$

106. *To find the conditions of parallelism of two planes, whose equations are given.*

Let the equations of the two planes be

$$\lambda x + \mu y + \nu z + \rho w = 0 \text{ and } \lambda' x + \mu' y + \nu' z + \rho' w = 0,$$

these planes intersect in the plane at an infinite distance, whose equation is

$$x + y + z + w = 0,$$

using tetrahedral coordinates.

From the three equations we deduce

$$(\lambda - \rho) x + (\mu - \rho) y + (\nu - \rho) z = 0,$$

$$(\lambda' - \rho') x + (\mu' - \rho') y + (\nu' - \rho') z = 0,$$

which are satisfied by an infinite number of values of the ratios $x : y : z$, they must therefore be identical equations;

$$\therefore \frac{\lambda - \rho}{\lambda' - \rho'} = \frac{\mu - \rho}{\mu' - \rho'} = \frac{\nu - \rho}{\nu' - \rho'},$$

these are the equations of condition, which may also be written

$$\begin{vmatrix} \lambda, & \mu, & \nu, & \rho \\ \lambda', & \mu', & \nu', & \rho' \\ 1, & 1, & 1, & 1 \end{vmatrix} = 0.$$

They also follow immediately from $p - p' = q - q' = \ldots$.

107. *To find the length of the perpendicular from a given point upon a plane given in four-plane or tetrahedral coordinates.*

Let the equation of the plane with four-plane coordinates be

$$\lambda x + \mu y + \nu z + \rho w = 0,$$

and let (x', y', z', w') be the given point.

Suppose the whole system to be referred to rectangular Cartesian coordinate axes, and let the equations of the four planes of reference be

$$l_1\xi + m_1\eta + n_1\zeta = p_1,$$
$$l_2\xi + m_2\eta + n_2\zeta = p_2 \text{ \&c.},$$

then the equation of the given plane will become

$$(\lambda l_1 + \mu l_2 + \nu l_3 + \rho l_4)\,\xi + \ldots = \lambda p_1 + \mu p_2 + \nu p_3 + \rho p_4,$$

hence, by Art. 70, the perpendicular required will be

$$\frac{\lambda x' + \mu y' + \nu z' + \rho w'}{\sigma} \qquad (1),$$

where $\sigma^2 = (\lambda l_1 + \mu l_2 + \nu l_3 + \rho l_4)^2 + \ldots$

$$= \lambda^2 (l_1^{\,2} + m_1^{\,2} + n_1^{\,2}) + \ldots + 2\lambda\mu\,(l_1 l_2 + m_1 m_2 + n_1 n_2) + \ldots$$
$$= \lambda^2 + \mu^2 + \nu^2 + \rho^2 - 2\lambda\mu \cos(AB) - \ldots \qquad (2),$$

(AB) being written for the dihedral angle between the faces of the tetrahedron opposite to A and B.

If p, q, r, s be the perpendiculars from A, B, C, D on the given plane, p being the perpendicular from $(p_0, 0, 0, 0)$, the expression (1) will give

$$p = \frac{\lambda p_0}{\sigma}, \quad q = \frac{\mu q_0}{\sigma}, \text{ \&c.}, \qquad (3),$$

whence the equation of the plane in four-plane coordinate will be

$$\frac{px}{p_0} + \frac{qy}{q_0} + \frac{rz}{r_0} + \frac{sw}{s_0} = 0\,;$$

and, in tetrahedral coordinates

$$px + qy + rz + sw = 0,$$

where, by equations (2) and (3),

$$\left(\frac{p}{p_0}\right)^2 + \left(\frac{q}{q_0}\right)^2 + \left(\frac{r}{r_0}\right)^2 + \left(\frac{s}{s_0}\right)^2 - 2\frac{pq}{p_0 q_0}\cos AB - \ldots\ldots = 1;$$

so that the left side of the equation in either form represents the perpendicular from any point (x, y, z, w).

108. The method which we have adopted in the last article shews that the quadric function $\lambda^2 + \mu^2 + \ldots - 2\lambda\mu\cos(AB) - \ldots$ is reducible by transformation to three squares, the condition of which is that the discriminant vanishes, or that

$$\begin{vmatrix} -1, & \cos(AB), & \cos(AC), & \cos(AD) \\ \cos(AB), & -1, & \cos(BC), & \cos(BD) \\ \cos(AC), & \cos(BC), & -1, & \cos(CD) \\ \cos(AD), & \cos(BD), & \cos(CD), & -1 \end{vmatrix} = 0.$$

Also, since each of the three squares is positive, there is only one system of values which reduces the function to zero, viz. that which belongs to a plane at an infinite distance, for which $p = q = r = s$, whence, by (3) Art. 107, $\lambda p_0 = \mu q_0 = \nu r_0 = \rho s_0$.

That the discriminant vanishes, may be shewn independently by projecting any three of the faces of the tetrahedron on the fourth, and obtaining the determinant from the four equations similar to

$$A - B\cos(AB) - C\cos(AC) - D\cos(AD) = 0.$$

109. *To find the angle between two planes whose equations are given.*

Let the equations of the planes be

$$\lambda x + \mu y + \nu z + \rho w = 0, \text{ and } \lambda' x + \mu' y + \nu' z + \rho' w = 0,$$

using the same method as in Art. (107), if θ be the angle between the planes

$$\sigma\sigma' \cos\theta = (\lambda l_1 + \mu l_2 + \nu l_3 + \rho l_4)(\mu' l_1 + \mu' l_2 + \nu' l_3 + \rho' l_4) + \ldots$$
$$= \lambda\lambda' + \mu\mu' + \nu\nu' + \rho\rho' - (\lambda\mu' + \lambda'\mu)\cos(AB) - \ldots.$$

110. *To find the direction-cosines of the normal to a plane whose equation is given in tetrahedral coordinates.*

Let $px + qy + rz + sw = 0$ be the given equation.

The equations of the perpendicular from A on the opposite face are easily shewn to be

$$\frac{x-1}{-\frac{1}{p_0}} = \frac{y}{\frac{\cos(AB)}{q_0}} = \frac{z}{\frac{\cos(AC)}{r_0}} = \frac{w}{\frac{\cos(AD)}{s_0}} = R,$$

therefore, where it meets the given plane,

$$p - R\left\{\frac{p}{p_0} - q\,\frac{\cos(AB)}{q_0} - r\,\frac{\cos(AC)}{r_0} - s\,\frac{\cos(AD)}{s_0}\right\} = 0,$$

$$\text{and } p = R\cos(p,\, p_0)\,;$$

$$\therefore\ \cos(p,\, p_0) = \frac{p}{p_0} - q\,\frac{\cos(AB)}{q_0} - r\,\frac{\cos(AC)}{r_0} - s\,\frac{\cos(AD)}{s_0},$$

and similar expressions for the other direction-cosines.

VI.

(1) Shew that for every point in a plane through the edge AB bisecting the angle between the planes CAB, DAB,

$z - w = 0$, if the angle be the internal angle,

$z + w = 0$, ... external ....

(2) Shew that for every point in a plane drawn through the vertex A parallel to the opposite face,

$$By + Cz + Dw = 0\,;$$

or, with tetrahedral coordinates,

$$y + z + w = 0.$$

(3) If AO be drawn perpendicular to the opposite face BCD, then for any point in AO,

$$\frac{By}{\triangle COD} = \frac{Cz}{\triangle DOB} = \frac{Dw}{\triangle BOC} = AO - x.$$

(4) Every point in a plane through CD parallel to AB satisfies the equation, in tetrahedral coordinates,

$$x + y = 0.$$

(5) A point is determined in tetrahedral coordinates by the equations $lx = my = nz = rw$.

What plane is represented by the equation $my = nz$, and what straight line by the equations $my = nz = rw$?

(6) At any point in the straight line joining the first points of trisection of AB and CD, the tetrahedral coordinates satisfy the equations

$$x = 2y,\ z = 2w.$$

(7) Shew that the coordinates (tetrahedral) of the centre of gravity of the fundamental tetrahedron are given by $x = y = z = w$.

(8) The three straight lines joining the middle points of opposite edges of a tetrahedron meet in its centre of gravity.

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(9) A plane cuts each of the six edges of a tetrahedron; another point is taken in each edge, so as to cut it harmonically; prove that the six planes through these latter points and the opposite edges of the tetrahedron intersect in one point.

(10) If the equations of a point O be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = \frac{w}{r},$$

and AO, BO, CO, DO be joined and produced to A', B', C', D', such that O bisects the lines AA', &c., the tetrahedral coordinates of the point A' will satisfy the equations

$$\frac{2x}{l - m - n - r} = \frac{y}{m} = \frac{z}{n} = \frac{w}{r} = \frac{2}{l + m + n + r},$$

and similarly for B', C', D'.

(11) The line joining the centres of the two spheres which touch the faces of the tetrahedron $ABCD$ opposite to A, B respectively, and the other faces produced, will intersect the edge CD in a point P, such that $CP : PD :: \Delta ACB : \Delta ADB$, and the edge AB (produced) in a point Q, such that $AQ : BQ :: \Delta CAD : \Delta CBD$.

(12) If two opposite edges of a tetrahedron be trisected, and the points of trisection be joined by two lines in either order, shew that the line which bisects these lines will also bisect two other opposite edges.

(13) l, l' are the lengths of two of the lines joining the middle points of opposite edges of a tetrahedon, ω the angle between these lines, a, a' those edges of the tetrahedron which are not met by either of the lines,

$$\cos\omega = \frac{a^2 - a'^2}{4ll'}.$$

(14) A point O is taken within a tetrahedron $ABCD$, so as to be the centre of gravity of the feet of the perpendiculars let fall from O on the faces; prove that the distances of O from the several faces are proportional respectively to those faces.

(15) Shew that the reciprocals of the radii of the spheres which can be drawn to touch the four faces of a tetrahedron, are the positive values of the expression

$$\pm\frac{1}{p_0}\pm\frac{1}{q_0}\pm\frac{1}{r_0}\pm\frac{1}{s_0},$$

p_0, q_0, r_0, s_0 being the perpendiculars from the angular points upon the opposite faces.

(16) Lines are drawn from the angular points of a tetrahedron, through the centre of the sphere circumscribing the tetrahedron, to meet the opposite faces; prove that the sum of their reciprocals is three times the reciprocal of the radius of the sphere.

(17) The inscribed sphere of a tetrahedron $ABCD$ touches the faces in A', B', C', D'; prove that AA', BB', CC', DD' will meet in a point, if

$$\cos\tfrac{1}{2}a\cos\tfrac{1}{2}\alpha=\cos\tfrac{1}{2}b\cos\tfrac{1}{2}\beta=\cos\tfrac{1}{2}c\cos\tfrac{1}{2}\gamma;$$

where a, α; b, β; c, γ are pairs of dihedral angles at opposite edges.

CHAPTER VII.

FOUR-POINT COORDINATE SYSTEM.
THE POINT. THE PLANE.

111. In the *Four-Plane Coordinate System*, the position of a point is given by its algebraical distances from four fundamental planes, given in position, which do not pass through one point, so that they form the plane faces of a tetrahedron of finite volume.

The position of a plane is given by a relation between the four-plane coordinates, which exists for every point which lies in that plane.

In the *Four-Point Coordinate System*, the position of a plane is given by its distances from four fundamental points, given in position, which do not all lie in one plane, so that they form the angular points of a tetrahedron of finite volume.

These distances are called *Point Coordinates* of the plane.

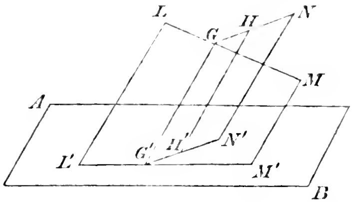
An infinite number of planes can be drawn through any given point, and it can be shewn that the point coordinates of each of these planes satisfy a linear equation; this equation determines the position of the point, and is called the equation of the point.

112. *To find the distance, estimated in any direction, of a point, whose position relative to fixed points is known, from a plane whose distances from the fixed points are given.*

Let L, M be two points, and let a point G be taken in the straight line joining them, such that $\lambda . LG = \mu . MG$.

Let LL', MM', GG' be parallel lines, drawn in a given direction, meeting a given plane AB in L', G', M', then it is evident that

$$\frac{LL' - GG'}{LG} = \frac{GG' - MM'}{MG},$$



$$\text{and } \lambda.(LL' - GG') = \mu.(GG' - MM');$$

$$\therefore \lambda.LL' + \mu.MM' = (\lambda + \mu)\, GG'. \qquad (1)$$

If N be any other point, and H be taken in GN so that $(\lambda + \mu)\, GH = \nu.HN$, and if HH', NN' be drawn parallel to LL',

$$\text{then } \nu.NN' + (\lambda + \mu)\, GG' = (\lambda + \mu + \nu)\, HH';$$

$$\therefore \lambda.LL' + \mu.MM' + \nu NN' = (\lambda + \mu + \nu)\, HH'. \qquad (2)$$

If there be four fundamental points L, M, N, R, and K be taken in HR, such that $(\lambda + \mu + \nu)\, HK = \rho.RK$, and RR', KK' be drawn parallel to LL' meeting the plane AB in R', K', then

$$\rho.RR' + (\lambda + \mu + \nu)\, HH' = (\lambda + \mu + \nu + \rho)\, KK';$$

$$\therefore \lambda.LL' + \mu.MM' + \nu.NN' + \rho.RR' = (\lambda + \mu + \nu + \rho)\, KK'. \quad (3)$$

Thus, the position of the point K relative to four fundamental points is given by the quantities λ, μ, ν, ρ, and it may be denoted by $(\lambda, \mu, \nu, \rho)$; and the distance of the point thus determined from a plane, estimated in any direction, is known by (3) in terms of the distances of the four fundamental points estimated in the same direction. The equations (1) and (2) determine the same thing for two and three fundamental points respectively.

113. If the formula for the position of the centre of gravity of any number of particles be assumed, the results of the preceding article will be obtained at once, by considering masses proportional to λ, μ, ν, ρ placed at the fundamental points, the point K (Art. 112) being the centre of gravity of these particles, where the masses may be supposed negative if necessary.

114. *To find the equation of a point in four-point coordinates.*

If the four points lie in a plane, then by the construction of Art. 112, it is obvious that the point K, being in the line HR, will lie in the same plane with the four points. This accounts for the restriction with respect to the fundamental points, that they shall not lie in one plane, because the equation obtained would then always denote a point in the same plane, and could not be the general equation of a point in space.

If A, B, C, D be any four points which do not lie in a plane, p, q, r, s the perpendicular distances of a plane from these points, estimated in a given direction, $(\lambda, \mu, \nu, \rho)$ a point P, with reference to these fundamental points, and t the perpendicular from P upon the plane; we shall have, by (3), Art. 112,

$$\lambda p + \mu q + \nu r + \rho s = (\lambda + \mu + \nu + \rho)\, t,$$

and $t = 0$ for every plane passing through P,

$$\therefore\ \lambda p + \mu q + \nu r + \rho s = 0.$$

Hence, upon the same principle, upon which, in the four-plane coordinates, x, y, z, w being the coordinates of any point,

$$\lambda x + \mu y + \nu z + \rho w = 0$$

is the equation of that plane, in which a series of points lie, whose coordinates satisfy the equation; so, p, q, r, s being the coordinates of a plane in the four-point system of coordinates, $\lambda p + \mu q + \nu r + \rho s = 0$ is the equation of that point, through which all planes pass whose coordinates satisfy the equation.

115. *To interpret the constants in the equation of a point.*

Let the equation of the point P be $\lambda p + \mu q + \nu r + \rho s = 0$; the four-point coordinates of BCD are p_0, 0, 0, 0, hence, the perpendicular on BCD from P is, by Art. 112, $\dfrac{\lambda p_0}{\lambda + \mu + \nu + \rho}$, therefore $\dfrac{\lambda}{\lambda + \mu + \nu + \rho}$ is the corresponding tetrahedral coordinate of P for the same fundamental tetrahedron; so that λ, μ, ν, ρ are proportional to the tetrahedral coordinates of the point whose equation is given, which may therefore be written as in Art. 101,

$$xp + yq + zr + ws = 0.$$

In this form the first member of the equation is the perpendicular from the point given by the equation, also denoted by (x, y, z, w), upon the plane whose four-point coordinates are p, q, r, s.

116. *To find the equation of a point which divides the straight line joining two points, whose equations are given, in a given ratio.*

Let the equations of the two points P, P' be

$$xp + yq + zr + ws = 0, \text{ and } x'p + y'q + z'r + w's = 0,$$

and let Q be a point in PP', such that

$$PQ : QP' :: m : l;$$

for every plane passing through Q the perpendiculars from P, P' are in the same ratio, and observing that these perpendiculars are drawn in opposite directions if Q be between P and P', we have

$$l(xp + yq + zr + ws) + m(x'p + y'q + z'r + w's) = 0,$$

the equation required.

117. *To find the equation of a point at an infinite distance.*

Let the equation of a point be $\lambda p + \mu q + \nu r + \rho s = 0$, the perpendicular distance from this point on any plane (p, q, r, s) is $\dfrac{\lambda p + \mu q + \nu r + \rho s}{\lambda + \mu + \nu + \rho}$.

If this point be at an infinite distance, we must have the condition $\lambda + \mu + \nu + \rho = 0$, which expresses that, if (x, y, z, w) be the point in tetrahedral coordinates, $x + y + z + w = 0$, or that the point lies in the plane at an infinite distance, (Art. 105).

118. The signs of the constants in the equation of a point in the general form $\lambda p + \mu q + \nu r + \rho s = 0$, in order that the point may lie in the different portions of space cut off by the indefinite planes which form the faces of the fundamental tetrahedron, can be obtained by considering that $\dfrac{\lambda}{\sigma}, \dfrac{\mu}{\sigma}, \dfrac{\nu}{\sigma}, \dfrac{\rho}{\sigma}$ are the tetrahedral coordinates of the point, if $\sigma = \lambda + \mu + \nu + \rho$.

119. *To find the distance between two points, whose four-point coordinates are given.*

The distance between two points P, P', whose equations are $\lambda p + \mu q + \nu r + \rho s = 0$, and $\lambda' p + \mu' q + \nu' r + \rho' s = 0$, can be found from Art. 96, by considering that $\frac{\lambda}{\sigma}, \frac{\mu}{\sigma}, \frac{\nu}{\sigma}, \frac{\rho}{\sigma}$, and $\frac{\lambda'}{\sigma'}, \frac{\mu'}{\sigma'}, \frac{\nu'}{\sigma'}, \frac{\rho'}{\sigma'}$ are tetrahedral coordinates of two points;

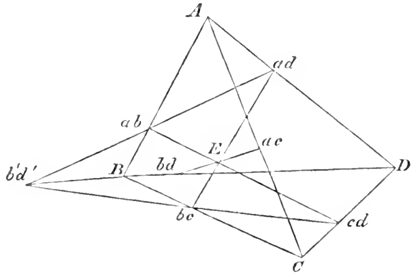
$$\therefore\ PP'^2 = \Sigma \left\{ \left(\frac{\lambda'}{\sigma'} - \frac{\lambda}{\sigma}\right) \left(\frac{\mu}{\sigma} - \frac{\mu'}{\sigma'}\right) AB^2 \right\}.$$

120. *To shew that the straight lines joining the middle points of opposite edges of a tetrahedron intersect and bisect each other.*

The equation of the middle point of AB is $p + q = 0$ (Art. 116), and of the middle point of CD is $r + s = 0$; therefore the equation of the middle point of the line joining these is $p + q + r + s = 0$, which for the same reason bisects the lines joining the middle points of the other opposite edges.

121. The student is recommended to examine carefully the processes employed in the following applications of point-coordinates.

Let $\lambda p + \mu q + \nu r + \rho s = 0$ be the equation of any point E, and let planes be drawn through this point and each of the



edges; and let (ab) denote the point in which the plane ECD meets AB, and similarly for the other edges.

The point E lies in the straight line joining (ab), for which $\lambda p + \mu q = 0$, and (cd), for which $\nu r + \rho s = 0$, since its equation is of the form $L(\lambda p + \mu q) + M(\nu r + \rho s) = 0$.

$$\begin{array}{lll} \text{Since for} & (ab), & \left.\begin{array}{l}\lambda p + \mu q = 0,\\ \lambda p + \rho s = 0,\end{array}\right\} \\ & (ad), & \\ & (bc), & \left.\begin{array}{l}\mu q + \nu r = 0,\\ \nu r + \rho s = 0,\end{array}\right\} \\ & (cd), & \end{array}$$

the straight lines joining these pairs of points meet BD in a point $(b'd')$ whose equation is $\mu q = \rho s$, and the equation of (bd) is $\mu q + \rho s = 0$; therefore $b'd'$, bd divide BD harmonically.

Similarly, the line joining (ab), (ac) and (bd), (cd) intersect BC in $(b'c')$, for which $\mu q = \nu r$; and the straight line (ab), (cd) meets the plane passing through A and the points $(b'c')$, $(c'd')$ in the point whose equation is

$$2\lambda p + 2\mu q - \nu r - \rho s = 0,$$

since this equation may be written

$$2\lambda p + (\mu q - \nu r) + (\mu q - \rho s) = 0.$$

Again, the equation $2\lambda p + \mu q + \nu r = 0$, being of the form

$$Lp + Ms + N(\lambda p + \mu q + \nu r + \rho s) = 0,$$

represents a point in the plane ADE, and, being of the form $L(\lambda p + \mu q) + M(\lambda p + \nu r) = 0$, the point lies on the line joining (ab), (ac), and is obviously in the plane ABC.

Let the straight lines AE, BE meet the opposite faces in A', B'; the equations of these points are

$$\mu q + \nu r + \rho s = 0, \text{ and } \lambda p + \nu r + \rho s = 0,$$

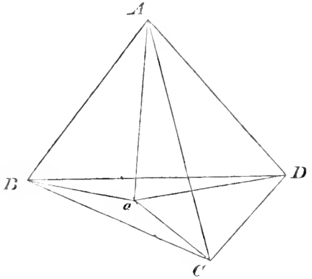
and therefore $A'B'$ intersects AB in the point $\lambda p - \mu q = 0$, the same point in which $(ac)(bc)$, $(ad)(bd)$, meet AB.

122. Many of the results of the following articles have been obtained in the preceding chapter, but the independent processes adopted here will serve to illustrate the subject on which we are engaged.

123. *To find the inclinations of a plane, whose coordinates are given, to the faces of the fundamental tetrahedron.*

Let p, q, r, s be the coordinates of a plane, λ, μ, ν, ρ the cosines of the angles between the direction in which the coordinates are measured, and the perpendiculars from A, B, C, D

on the opposite faces, and let $Aa = p_0$ be the perpendicular from A on BCD.



The tetrahedral coordinates of a are

$$x = 0,\ y = \frac{aACD}{BACD} = \frac{aCD}{BCD} = \frac{B\cos(AB)}{A} = \frac{p_0}{q_0}\cos(AB),$$

$$z = \frac{p_0}{r_0}\cos(AC),\quad w = \frac{p_0}{s_0}\cos(AD).$$

The equation of the point a is therefore

$$\frac{p_0}{q_0}\cos(AB)\,q + \frac{p_0}{r_0}\cos(AC)\,r + \frac{p_0}{s_0}\cos(AD)\,s = 0,$$

and, the coefficients being tetrahedral coordinates, the first member is the perpendicular from a on the plane (p, q, r, s), and therefore $= p - p_0\lambda$,

$$\therefore\ \lambda = \frac{p}{p_0} - \frac{q}{q_0}\cos(AB) - \frac{r}{r_0}\cos(AC) - \frac{s}{s_0}\cos(AD)\text{; see Art. 110.}$$

Similar values may be obtained for μ, ν, ρ.

124. *To find the relation between the four-point coordinates of a plane.*

The tetrahedral coordinates of P, the foot of the perpendicular from A on the plane (p, q, r, s), are

$$\frac{1}{p_0}(p_0 - p\lambda),\ -\frac{p\mu}{q_0},\ -\frac{p\nu}{r_0},\ -\frac{p\rho}{s_0};$$

therefore, from the equation of the point P, we have

$$\frac{1}{p_0}(p\lambda - p_0)\, p + \frac{p\mu}{q_0}\, q + \frac{p\nu}{r_0}\, r + \frac{p\rho}{s_0}\, s = 0,$$

$$\therefore\ \frac{p}{p_0}\lambda + \frac{q}{q_0}\,\mu + \frac{r}{r_0}\,\nu + \frac{s}{s_0}\,\rho = 1.$$

Hence, substituting the values of λ, μ, ν, ρ found above, we obtain the relation between the coordinates

$$\frac{p^2}{p_0^{\,2}} + \frac{q^2}{q_0^{\,2}} + \frac{r^2}{r_0^{\,2}} + \frac{s^2}{s_0^{\,2}} - \frac{2pq}{p_0 q_0}\cos(AB) - \ldots = 1, \text{ see Art. 107.}$$

125. If the plane be at an infinite distance the left side of the above equation will vanish, the only system of values for which this will be the case being $p = q = r = s$, (Art. 108).

Since the coordinates are equal and of infinite magnitude, the expression for λ in Art. 123 gives

$$0 = \frac{1}{p_0} - \frac{1}{q_0}\cos(AB) - \frac{1}{r_0}\cos(AC) - \frac{1}{s_0}\cos(AD),$$

$$\text{or }\ 0 = A - B\cos(AB) - C\cos(AC) - D\cos(AD).$$

We have also from Art. 124

$$\frac{\lambda}{p_0} + \frac{\mu}{q_0} + \frac{\nu}{r_0} + \frac{\rho}{s_0} = 0,$$

a linear relation between the direction cosines of any plane.

126. The equation which connects the four-point coordinates of a plane is of the second degree, whereas the corresponding equation for four-plane or tetrahedral coordinates is of the first degree.

The reason of this equation being of the second degree should be explained.

If three four-point coordinates of a plane be given, suppose p, q, r, this plane must touch three spheres whose radii are p, q, r, centres A, B, C; and, if we suppose the most general case, there will be eight such planes, two for which the three spheres lie on the same side, in which cases p, q, r will be of the same sign, and six for which one of the spheres lies on the opposite side to the other two, in which cases two of the coordinates p, q, r will be of opposite sign to the third.

Whether p, q, r be of the same sign, or p be of opposite sign to q and r, there are two positions of the touching plane; that is, there are two values of s, viz. the perpendiculars in these two positions; the equation must, therefore, be of the second degree in s, and similarly for p, q, and r.

Hence, although, when three tetrahedral coordinates are given, the fourth is fully determined by the equation of condition; this is not the case in four-point coordinates.

127. *To find the angle between two planes whose coordinates are given.*

Let (p', q', r', s') and (p'', q'', r'', s'') be two given planes denoted by U', U'', and let the line drawn perpendicular to U' from the fundamental point A meet U'' in the point N, so that $AN = \varpi = p'' \sec(U', U'')$.

Let λ', μ', ν', ρ' be the cosines of the angles between the normal to the plane U', and the normals to the corresponding faces of the fundamental tetrahedron.

The tetrahedral coordinates of N are

$$\frac{p_0 - \varpi\lambda'}{p_0}, \quad \frac{-\varpi\mu'}{q_0}, \quad \frac{-\varpi\nu'}{r_0}, \quad \frac{-\varpi\rho'}{s_0};$$

hence the equation of N is

$$\frac{p_0 - \varpi\lambda'}{p_0} p - \frac{\varpi\mu'}{q_0} q - \frac{\varpi\nu'}{r_0} r - \frac{\varpi\rho'}{s_0} s = 0;$$

and, since the plane U'' is a particular plane through N, we have, since $\dfrac{p''}{\varpi} = \cos(U', U'')$,

$$\cos(U', U'') = \frac{\lambda' p''}{p_0} + \frac{\mu' q''}{q_0} + \frac{\nu' r''}{r_0} + \frac{\rho' s''}{s_0}.$$

But $\lambda' = \dfrac{p'}{p_0} - \dfrac{q'}{q_0} \cos(AB) - \dfrac{r'}{r_0} \cos(AC) - \dfrac{s'}{s_0} \cos(AD)$ (Art. 123);

$$\therefore \cos(U', U'') = \frac{p'p''}{p_0^2} + \frac{q'q''}{q_0^2} + \ldots$$

$$- \frac{p'q'' + p''q'}{p_0 q_0} \cos(AB) - \ldots, \text{ see Art. 109.}$$

If the planes be parallel, since the perpendicular distance between them is constant, $p'-p''=q'-q''=r'-r''=s'-s''$.

As in Arts. 68 and 22 in the corresponding case with Cartesian coordinates, these conditions can be deduced from the value of $\cos(U', U'')$. For, since $(U', U'')=0$,

$$2\cos(U', U'')=2=\frac{p'^2}{p_0^2}+\ldots-\frac{2p'q'}{p_0q_0}\cos(AB)-\ldots$$

$$+\frac{p''^2}{p_0^2}+\ldots-\frac{2p''q''}{p_0q_0}\cos(AB)-\ldots \text{ (Art. 124)},$$

$$\text{whence } \frac{(p'-p'')^2}{p_0^2}+\ldots-\frac{2(p'-p'')(q'-q'')}{p_0q_0}\cos(AB)-\ldots=0.$$

But (Art. 108) this is only possible when

$$p'-p''=q'-q''=\ldots.$$

128. *To find the coordinates of a plane which passes through the intersections of two planes whose coordinates are given.*

Let (p', q', r', s') (p'', q'', r'', s'') be the planes U' and U'', and let (p, q, r, s) be the plane V passing through their line of intersection. The perpendiculars from the fundamental point A on the three planes all lie in a plane, and the relation between these three may be found from the trilinear coordinates corresponding to an evanescent fundamental triangle, whose angles are the angles between the planes, or the supplement of those angles; hence

$$p\sin(U', U'')=p'\sin(U'', V)+p''\sin(U', V);$$

$$\therefore p=lp'+mp'', \text{ where } l^2\pm 2lm\cos(U', U'')+m^2=1,$$

and similarly for q, r, s.

129. If $Lp+Mq+Nr+Rs=0$ be an equation involving one variable t in the first degree, it may be considered as the equation of a line, since it may be put into the form $u+tv=0$, and by varying t we may obtain every point in the line joining the points $u=0$, $v=0$.

130. If $Lp+Mq+Nr+Rs=0$ be an equation involving two variables t, t' in the first degree, it may be considered as

the equation of a plane, since it may be put in the form $u + tv + t'w = 0$, and by varying t, t' we may obtain every point in the plane passing through the three points $u = 0$, $v = 0$, and $w = 0$.

VII.

(1) The equation of the centre of gravity of the face ABC is

$$p + q + r = 0.$$

Hence, shew that the lines joining the vertices with the centres of gravity of the opposite faces meet in a point.

(2) The equation of the centre of the circle circumscribing the triangle ABC is

$$p \sin 2A + q \sin 2B + r \sin 2C = 0.$$

(3) The coordinates of the plane passing through the centres of gravity of the faces ACD, ADB, and ABC, are given by the equations

$$-\frac{p}{2} = q = r = s = \frac{p_0}{3}.$$

(4) If P be any point in BD, Q, R points in AC, such that

$$AQ : QC :: DP : PB :: CR : RA,$$

then PQ and PR will intersect the lines joining the middle points of BC, AD, and AB, CD respectively, and divide them in the same ratio as AC.

(5) If through the middle points of the edges BC, CD, DB straight lines be drawn parallel respectively to the opposite edges, these straight lines will meet in a point; and the line joining this point with A will pass through the centre of gravity of the pyramid.

(6) The equation of the centre of gravity of the surface of the tetrahedron is

$$(A + B + C + D)(p + q + r + s) = Ap + Bq + Cr + Ds.$$

(7) Shew that the equation of the centres of the eight spheres which touch the faces or the faces produced of the fundamental tetrahedron are

$$Ap \pm Bq \pm Cr \pm Ds = 0.$$

(8) The centre C of the inscribed sphere lies on the line joining G, H the centres of gravity of the volume and of the surface of the tetrahedron; also, shew that $CG = 3GH$.

(9) The points B, C, D are joined to the centres of gravity of the opposite faces, and the joining lines produced to points b, c, d, so that B, b, &c., are equidistant from the corresponding faces, prove that the coordinates of the plane bcd are given by the equations

$$-2p = q = r = s,$$

and that this plane divides the edges AB, AC, AD in the ratio 1 : 2.

(10) If points be taken in the lines joining B, C, D to the centres of gravity of the opposite faces, dividing them in the ratio $m : n$, the plane containing these points will divide the edges AB, AC, AD in the ratio $m : 2m + 3n$.

(11) If through any point P straight lines AP, BP, CP, DP be drawn meeting the opposite faces in a, b, c, d, the straight lines AB, ab will intersect, and their point of intersection and the point in which Cd meets AB will divide AB harmonically.

(12) The straight lines joining D to the intersection of AB, ab, and A to the intersection of DB, db, will intersect in a point lying on Bc.

(13) If through any point three straight lines be drawn, each meeting two opposite edges of a tetrahedron $ABCD$; and if a, α; b, β; c, γ be the points where these straight lines meet the edges BC, AD; CA, BD; AB, CD; then will

$$Ba \, . \, C\gamma \, . \, D\beta = B\beta \, . \, Ca \, . \, D\gamma,$$

$$Cb \, . \, A\alpha \, . \, D\gamma = C\gamma \, . \, Ab \, . \, D\alpha,$$

$$Ac \, . \, B\beta \, . \, D\alpha = A\alpha \, . \, Bc \, . \, D\beta,$$

$$Ab \, . \, Bc \, . \, Ca = Ac \, . \, Ba \, . \, Cb.$$

CHAPTER VIII.

LOCI OF EQUATIONS. TANGENTIAL EQUATIONS OF SURFACES. TORSES. DUAL INTERPRETATION OF EQUATIONS. BOOTHIAN COORDINATES.

Tangential Equations of Surfaces and Torses.

131. If p, q, r, s be the coordinates of a plane referred to a four-point system, to find what is represented by the general equation $F(p, q, r, s) = 0$, we observe that there are generally an infinite number of planes, the coordinates of each of which satisfy the equation, and that these planes envelope a surface of which the equation is called the tangential equation, from the circumstance that each plane of the system is a tangent plane to the surface, and the surface may be called the envelope locus of the equation.

As in a quadriplanar or tetrahedral system, if we take three points on the locus of $\phi(x, y, z, w) = 0$, the plane containing these three points will ultimately be a tangent plane to the locus, if the three points become ultimately coincident, so, if we take three planes touching the envelope locus of $F(p, q, r, s) = 0$, the point in which these three planes intersect will be ultimately the point of contact of these planes when they become coincident.

Thus, $p = f$ is the tangential equation of a sphere whose centre is the fundamental point A.

In the case of the linear equation $\lambda p + \mu q + \nu r + \rho s = 0$, the envelope degenerates into a point, through which all the planes pass which correspond to the various solutions of the equation.

132. Again, if we take two surfaces represented in the tetrahedral system by $\phi(x, y, z, w) = 0$, and $\phi_1(x, y, z, w) = 0$, the coordinates of every point in their curve of intersection will satisfy both equations, and the two equations will determine the

position of this curve; so if $F(p, q, r, s) = 0$, and $F_1(p, q, r, s) = 0$ represent two surfaces in the four-point system, the coordinates of every plane which touches both surfaces satisfy the two equations, and the series of planes so determined have for their envelope a particular kind of surface, called a *Developable Surface*, or, according to Cayley, a *Torse*, which touches the two envelope loci of the equations, the reason of the term developable being as follows:

If we take three consecutive planes P, Q, R, each of which touches the two loci S, S_1, of the equations $F(p, q, r, s) = 0$, $F_1(p, q, r, s) = 0$, Q will be intersected by P and R in two straight lines (P, Q), and (Q, R), and in the limit the portion of Q intercepted between these lines will ultimately be a portion of the envelope of the planes; similar portions of P and R will, with the former, constitute three elements of the envelope, each of which will touch both S and S_1; and this envelope is called a developable surface, because the three elements can be developed into one plane by turning them about the lines (P, Q), (Q, R); and the same is true for all the elements.

According to this interpretation, the developable surface touching two spheres, $p = f$, $q = g$, is a system of two cones, of which one will be imaginary if the spheres intersect.

133. It may appear, from what has been said, that, since two equations, as well as one, in the four-point system determine a surface, the case of this system is not analogous to that of the three or four-plane system. But the two kinds of surfaces in the point system are really as distinct from one another as the surface and curve in the plane system.

For, as in the four-plane system one equation represents the limitation that the current point must remain on a surface, and the second equation confines the motion on that surface to positions such that it also remains on a second surface; so in the four-point system one equation limits the motion of a plane to positions in which it touches a surface, and the second equation allows the plane to touch that surface only in such a manner that it also touches a second surface.

Stated in this way the analogy is complete, the point moving

in the direction of a line, the plane turning round a line, to gain the consecutive position.

134. As a simple example of the use of the tangential equation of a surface, we will consider the properties of the *Poles of Similitude* of four spheres; the poles of similitude of two spheres being the vertices of the two cones which envelope both spheres, points from which the lengths of the tangents to the spheres are proportional to their radii.

These poles of similitude are called *internal* or *external* poles, according as they lie on the line joining the centres, or on this line produced.

135. *To find the relative positions of the internal or external poles of similitude of four spheres.*

Let the centres of the spheres be taken for the fundamental points A, B, C, and D, and let their radii be r_1, r_2, r_3, r_4.

The tangential equations of the spheres are $p = r_1$, $q = r_2$, $r = r_3$, $s = r_4$.

The external and internal poles of similitude of the spheres (A) and (B) have equations

$$\frac{p}{r_1} \mp \frac{q}{r_2} = 0, \text{ and similarly for the rest.}$$

(1) The external poles of (AB), (AC), and (AD) lie in a plane whose coordinates are connected by the equations

$$\frac{p}{r_1} = \frac{q}{r_2} = \frac{r}{r_3} = \frac{s}{r_4},$$

which evidently contains also the external poles of (BC), (CD) and (DB).

(2) The coordinates of the plane containing the external poles of (AB) and (AC) and the internal pole of (AD) satisfy the equations

$$\frac{p}{r_1} = \frac{q}{r_2} = \frac{r}{r_3} = -\frac{s}{r_4},$$

and the same plane evidently contains the external pole of (BC) and the internal poles of (BD) and (CD).

(3) The coordinates of the plane containing the external pole of (AB) and the internal poles of (AC) and (AD) satisfy the equations

$$\frac{p}{r_1} = \frac{q}{r_2} = -\frac{r}{r_3} = -\frac{s}{r_4},$$

and this plane evidently contains the external pole of (CD) and the internal poles of (BC) and (BD).

Hence one plane contains the six external poles, four planes contain each three external and three internal poles, and three contain each two external and four internal poles.

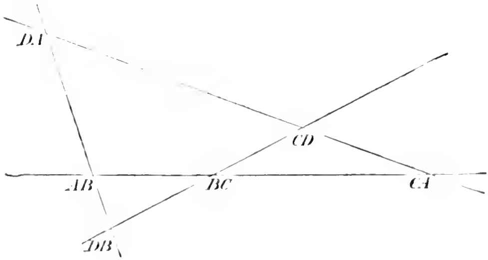
The poles of similitude lie in eight planes, which are called planes of similitude, each of which passes through six poles of similitude situated three and three in four straight lines.

Thus for the six external poles

$$\frac{p}{r_1} = \frac{q}{r_2} = \frac{r}{r_3} = \frac{s}{r_4},$$

$$\text{and } \frac{q}{r_2} - \frac{r}{r_3} = \frac{p}{r_1} - \frac{r}{r_3} - \left(\frac{p}{r_1} - \frac{q}{r_2}\right) = 0;$$

therefore the external pole of (BC) lies in the line joining those of (AB) and (AC).



Similarly it lies in the lines joining those of (DB) and (DC).

Hence, the six external poles lie in the sides of a plane quadrilateral, as in the figure.

Dual Interpretations of Equations.

136. By what has preceded, we see that all homogeneous equations and systems of two equations in four variables α, β, γ, δ, admit of a *dual* interpretation, according as we conceive the four variables to be tetrahedral or four-point coordinates.

Thus $\lambda\alpha + \mu\beta + \nu\gamma + \rho\delta = 0$ is the equation, in these two methods of viewing it, of a plane or of a point.

So if α, β, γ, δ be tetrahedral coordinates, the equation $F(\alpha, \beta, \gamma, \delta) = 0$ gives a surface on which every point lies, whose tetrahedral coordinates satisfy the equation, while if they be point coordinates, the equation gives a surface touched by every plane whose coordinates satisfy the equation.

Two such equations may in like manner be interpreted to define (1) a curve, which is the intersection of the two surfaces represented by the separate equations; or (2) a torse enveloped by all planes which touch both surfaces represented by the separate equations.

Thus, the dual results given by the method of Reciprocal Polars, which will be seen to apply to three as well as to two dimensions, may be obtained by giving this dual interpretations to all our equations.

We cannot, however, in a volume of moderate compass, pretend to include all the dual results to which our equations might give rise, but must confine ourselves to a development of the methods most generally useful.

Boothian Coordinates.

137. There is another system of coordinates which bears the same relation to four-point coordinates as the Cartesian to the tetrahedral system; these coordinates have been called Boothian from Dr. Booth, who first published the method in Dublin.*

We have seen that the equation of a plane in one form is $\alpha x + \beta y + \gamma z = 1$, where α, β, γ are the reciprocals of the inter-

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* Both Chasles and Plücker seem to have conceived the idea previously. Briot and Bouquet, *Geom. Anal.* p. 388.

cepts on the coordinate axes, if this plane pass through a point P, (f, g, h), then $\alpha f + \beta g + \gamma h = 1$ is an equation between α, β, and γ, which will be true for all planes passing through P. α, β, γ are called Boothian coordinates of a plane, and any equation of the first degree in α, β, γ expresses that the plane passes through a certain fixed point, and may be considered the equation of that point.

Any equation whatever between α, β, γ will express that the plane touches a certain fixed surface, and may be considered the equation of that surface.

Thus, we know that the equation of a sphere is $x^2 + y^2 + z^2 = r^2$, and that the distance of the plane $lx + my + nz = r$ from the origin is r; the plane therefore touches the sphere, and if the equation of the plane be written $\alpha x + \beta y + \gamma z = 1$, then we have $\alpha^2 + \beta^2 + \gamma^2 = \dfrac{l^2 + m^2 + n^2}{r^2} = \dfrac{1}{r^2}$, which is the Boothian equation of the sphere and is of the same form as the Cartesian.

138. *Cartesian coordinates are a particular case of tetrahedral, and Boothian of four-point coordinates.*

If we imagine the plane ABC of the tetrahedron of reference to move off to infinity and make the corresponding changes in our equations, any equation between x, y, z, w will become one between ξ, η, ζ ordinary Cartesian coordinates, and any equation between p, q, r, s will become one between α, β, γ Boothian coordinates of a plane.

Thus take the equation $px + qy + rz + sw = 0$, where (x, y, z, w) are tetrahedral coordinates of any point, and (p, q, r, s) four-point coordinates of any plane through (x, y, z, w). If the plane meet DA, DB, DC in a, b, c, and ξ, η, ζ be the Cartesian coordinates of (x, y, z, w) referred to the planes meeting in D,

then $x = \dfrac{\xi}{DA}$, $y = \dfrac{\eta}{DB}$, $z = \dfrac{\zeta}{DC}$, $w = 1 - \dfrac{\xi}{DA} - \dfrac{\eta}{DB} - \dfrac{\zeta}{DC}$,

whence the equation becomes

$$\frac{\xi}{DA} \cdot \frac{(s-p)}{s} + \frac{\eta}{DB} \cdot \frac{(s-q)}{s} + \frac{\zeta}{DC} \cdot \frac{s-r}{s} = 1.$$

$$\text{But } \frac{p}{s} = \frac{Aa}{Da}, \text{ or } \frac{s-p}{s} = \frac{DA}{Da}, \text{ or } \frac{s-p}{s\,.\,DA} = \frac{1}{Da} = \alpha,$$

and the equation of the plane becomes $\alpha\xi + \beta\eta + \gamma\zeta = 1$.

139. Any equation in which (x, y, z, w) are involved will generally have the coefficients of the different terms functions of DA, BC, ... edges of the tetrahedron of reference, so that although for any finite point (ξ, η, ζ) we have, when ABC moves off to infinity, $x = 0$, $y = 0$, $z = 0$, $w = 1$, yet we shall get a limiting equation between ξ, η, ζ when we have made the substitutions above and take DA, DB, DC, ... all infinite. So, for any equation in p, q, r, s, the coordinates of a plane at a finite distance from D, although in the limit $\frac{s}{p}, \frac{s}{q}, \frac{s}{r}$ are all equal and zero, yet by making

$$p = s\,(1 - \alpha\,.\,DA), \quad q = s\,(1 - \beta\,.\,DB), \quad r = s\,(1 - \gamma\,.\,DC),$$

we shall obtain a finite resulting equation in α, β, γ. But such transformation is very seldom of much advantage. It is, however, frequently convenient to render Cartesian or Boothian equations homogeneous by multiplying such terms as require it by w, w^2, ... or by δ, δ^2, ..., these being at some subsequent stage put equal to unity.

VIII.

(1) State some properties of the loci of the following equations, whether α, β, γ, δ be regarded as tetrahedral or four-point coordinates.

(1) $l\alpha\beta = m\gamma\delta$.

(2) $(\alpha + \beta)^2 = n\gamma\delta$.

(3) $\dfrac{l}{\alpha} + \dfrac{m}{\beta} + \dfrac{n}{\gamma} + \dfrac{r}{\delta} = 0$.

(2) If $u = 0$ be the equation of a surface of the second degree, $v = 0$ that of a plane, referred to tetrahedral coordinates, $v^2 = Au$ is the equation of a surface touching the surface $u = 0$ along the section made by $v = 0$.

Give the interpretation when the tetrahedral are replaced by four-point coordinates.

(3) Prove that if the fundamental points be in the angles of a regular tetrahedron, the tangential equations of the spheres circumscribing and inscribed in the tetrahedron will be respectively

$$p^2 + q^2 + r^2 + s^2 - qr - rp - pq - sp - sq - sr = 0,$$

$$\text{and } qr + rp + pq + sp + sq + sr = 0.$$

(4) Shew that in the same case the envelope locus of the equation

$$p^2 + q^2 + r^2 + s^2 - 2qr - 2rp - 2pq - 2sp - 2sq - 2sr = 0$$

will touch each of the edges of the fundamental tetrahedron.

(5) In the last problem, find the two tangent planes parallel to one of the faces of the tetrahedron, and shew that their distances from that face will be $\frac{a}{2\sqrt{(6)}}\{\sqrt{(3)} \pm 1\}$, a being the length of an edge.

(6) If two surfaces given by tetrahedral coordinates intersect in two plane curves, what is the corresponding property of the torse in the dual interpretation?

(7) Prove that the Boothian equation of a sphere passing through the origin of rectangular axes is of the form

$$\alpha^2 + \beta^2 + \gamma^2 = \left(\frac{1}{r} - l\alpha - m\beta - n\gamma\right)^2,$$

l, m, n being the direction-cosines of the radius r drawn through the origin.

(8) Two spheres of radii r, s, pass through the origin, and have their centres on the axes of x and y respectively; shew that the torse or developable surface enveloping both is a cone whose vertex has the Boothian equation

$$\alpha - \beta = \frac{1}{r} - \frac{1}{s}.$$

CHAPTER IX.

TRANSFORMATION OF COORDINATES.

140. The investigation of the properties of a surface represented by a given equation is often rendered more convenient by referring it to a different system of coordinate axes, in the choice of which we must be guided by the nature of the investigation proposed.

We proceed to obtain the working formulæ by which such a transformation may be effected.

141. *To change the origin of coordinates from one point to another, without altering the direction of the axes.*

Let (f, g, h) be the new origin referred to the primary system. If (x, y, z), (x', y', z') represent the position of the same point P referred to the first and second systems respectively, since the algebraical distance of the plane of $y'z'$ from that of yz is f, and x, x' are the algebraical distances of P from the planes of yz and $y'z'$, we have

$$x = x' + f,$$

and similarly, $$y = y' + g,$$

$$z = z' + h\,;$$

or, suppressing the accents, the transformation is effected by writing $x+f$, $y+g$, $z+h$, for x, y, z.

142. Since the formulæ thus obtained involve three arbitrary constants, we can generally by this transformation make the coefficients of three terms in the resulting equation vanish, but, since the coefficients of the terms of highest dimensions are unaltered, none of the three terms, so eliminated, can be of the same dimensions as the degree of the equation. Thus, in an equation of the second degree, we can generally destroy

the terms of one dimension in x, y, z; in an equation of the third degree three of the terms of two dimensions, and so on with equations of higher degrees. If, however, the terms, whose coefficients we desire to destroy, differ by more than one dimension from the degree of the equation, the equations for determining f, g, h in order to effect this result will rise to second, third, or higher degrees. For, if $F(x, y, z) = 0$ be an equation of the n^{th} degree, the transformed equation will be $F(x+f, y+g, z+h) = 0$, and the terms of the $(n-r)^{\text{th}}$ degree in the expansion can be represented in the form

$$\frac{1}{\lfloor n-r}\left(x\frac{d}{df}+y\frac{d}{dg}+z\frac{d}{dh}\right)^{n-r} F(f, g, h),$$

the coefficients of any term in which will involve $F(f, g, h)$ differentiated $n-r$ times with respect to the quantities f, g, h; hence the resulting equation for destroying any such term will be of the r^{th} degree. If three terms are to be destroyed, it is necessary that the three corresponding equations should be consistent; it may happen that these equations are not independent, in which case if two terms are made to disappear the third term will disappear at the same time, and we shall be able to get rid of a fourth term.

143. *To transform from one system of coordinates to another system having the same origin, both systems being rectangular.*

Let Ox, Oy, Oz be the first system, Ox', Oy', Oz' the second; let a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3 be the direction-cosines of Ox', Oy', Oz', referred to Ox, Oy, Oz; and x, y, z; x', y', z' coordinates of the same point in the two systems.

Then the algebraic distance of the point from the plane of yz is x; but, projecting the broken line $x'+y'+z'$, this same distance is $a_1x' + a_2y' + a_3z'$. Hence

$$\text{and similarly, }\left.\begin{aligned} x &= a_1x' + a_2y' + a_3z', \\ y &= b_1x' + b_2y' + b_3z', \\ z &= c_1x' + c_2y' + c_3z', \end{aligned}\right\} \qquad (1)$$

the formulæ required.

The nine constants introduced in these results are connected by six equations of condition, expressing that the two systems of

coordinates are rectangular, for since Ox, Oy, Oz are each two at right angles, we have the system of equations

$$\left.\begin{aligned} a_1^2 + b_1^2 + c_1^2 = 1, \\ a_2^2 + b_2^2 + c_2^2 = 1, \\ a_3^2 + b_3^2 + c_3^2 = 1, \end{aligned}\right\} \qquad (A)$$

and by reason of Ox', Oy', Oz' being also at right angles, the system

$$\left.\begin{aligned} a_2a_3 + b_2b_3 + c_2c_3 = 0, \\ a_3a_1 + b_3b_1 + c_3c_1 = 0, \\ a_1a_2 + b_1b_2 + c_1c_2 = 0. \end{aligned}\right\} \qquad (B)$$

The number of disposable constants in this transformation is therefore only three.

The relations (A), (B) subsisting among the nine constants involved in these formulæ may be replaced by

$$\left.\begin{aligned} a_1^2 + a_2^2 + a_3^2 = 1, \\ b_1^2 + b_2^2 + b_3^2 = 1, \\ c_1^2 + c_2^2 + c_3^2 = 1, \end{aligned}\right\} \qquad (A')$$

$$\left.\begin{aligned} b_1c_1 + b_2c_2 + b_3c_3 = 0, \\ c_1a_1 + c_2a_2 + c_3a_3 = 0, \\ a_1b_1 + a_2b_2 + a_3b_3 = 0. \end{aligned}\right\} \qquad (B')$$

if we consider Ox', Oy', Oz' the primary system of axes, in which case the direction-cosines of Ox, Oy, Oz, will be a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3. The equations (A') and (B') obtained from the same facts as the equations (A) and (B), are of course deducible from them. Either system may be obtained from the identical equation $x^2 + y^2 + z^2 \equiv x'^2 + y'^2 + z'^2$, by substituting for x, y, z their equivalents given in equations (1), or similarly for x', y', z'.

144. The relations between these constants may also be expressed in the following convenient form.

From the equations

$$a_1a_2 + b_1b_2 + c_1c_2 = 0, \quad a_3a_1 + b_3b_1 + c_3c_1 = 0,$$

we obtain immediately

$$\frac{a_1}{b_2c_3 - b_3c_2} = \frac{b_1}{c_2a_3 - c_3a_2} = \frac{c_1}{a_2b_3 - a_3b_2};$$

each member of these equations is therefore equal to

$$\frac{(a_1^2+b_1^2+c_1^2)^{\frac{1}{2}}}{\{(b_2c_3-b_3c_2)^2+(c_2a_3-c_3a_2)^2+(a_2b_3-a_3b_2)^2\}^{\frac{1}{2}}}$$

$$=\frac{(a_1^2+b_1^2+c_1^2)^{\frac{1}{2}}}{\{(a_2^2+b_2^2+c_2^2)(a_3^2+b_3^2+c_3^2)-(a_2a_3+b_2b_3+c_2c_3)^2\}^{\frac{1}{2}}}=\pm 1,$$

by equations (A), (B).

In a similar manner, we obtain

$$\frac{a_2}{b_3c_1-b_1c_3}=\frac{b_2}{c_3a_1-c_1a_3}=\frac{c_2}{a_3b_1-a_1b_3}=\pm 1,$$

$$\frac{a_3}{b_1c_2-b_2c_1}=\frac{b_3}{c_1a_2-c_2a_1}=\frac{c_3}{a_1b_2-a_2b_1}=\pm 1.$$

By using the equations (B') in a similar manner, we obtain

$$\frac{a_1}{b_2c_3-b_3c_2}=\frac{a_2}{b_3c_1-b_1c_3}=\frac{a_3}{b_1c_2-b_2c_1},$$

which shews that the ambiguities in the three systems of equations, here obtained, must be taken all of the same sign.

Any two of these three systems of equations may be taken as completely expressing the relations between the nine constants: the third system being immediately deducible from the other two.

We give the following problem as an illustration of the use which may be made of this transformation of coordinates.

145. *To find the equations of the straight lines which bisect the angles between two straight lines given by the equations*

$$lx+my+nz=0 \text{ and } ax^2+by^2+cz^2=0.$$

Choosing the axes, so that the axes of x' and y' shall be the two bisecting lines, and the plane of $x'y'$ shall contain the given lines, their equations will be of the form

$$z'=0, \text{ and } \lambda^2x'^2-\mu^2y'^2=0. \qquad (1)$$

The formulæ of transformation give

$$\begin{aligned} x' &= \alpha x + \beta y + \gamma z, \\ y' &= \alpha' x + \beta' y + \gamma' z, \\ z' &= lx + my + nz; \end{aligned}$$

$$\text{or } x = \alpha x' + \alpha' y' + lz',$$
$$y = \beta x' + \beta' y' + mz',$$
$$z = \gamma x' + \gamma' y' + nz';$$

hence, by (1), the second given equation becomes

$$a(\alpha x' + \alpha' y')^2 + b(\beta x' + \beta' y')^2 + c(\gamma x' + \gamma' y')^2 = 0,$$

which must be identical with $\lambda^2 x'^2 - \mu^2 y'^2 = 0$;

$$\therefore\ a\alpha\alpha' + b\beta\beta' + c\gamma\gamma' = 0; \tag{2}$$

$$\text{also } \alpha\alpha' + \beta\beta' + \gamma\gamma' = 0,$$

since the bisecting lines are at right angles;

$$\therefore\ \frac{\alpha\alpha'}{b-c} = \frac{\beta\beta'}{c-a} = \frac{\gamma\gamma'}{a-b}.$$

From these equations a variety of forms may be obtained for the equations of the bisecting lines; thus, if we require the forms $Ayz + Bzx + Cxy = 0$, and $lx + my + nz = 0$, since the first equation must reduce to the form $x'y' = 0$, when $z' = 0$, we obtain the relations

$$A\beta\gamma + B\gamma\alpha + C\alpha\beta = 0, \text{ and } A\beta'\gamma' + B\gamma'\alpha' + C\alpha'\beta' = 0;$$

$$\therefore\ \frac{A}{\alpha\alpha'(\beta\gamma' - \beta'\gamma)} = \frac{B}{\beta\beta'(\gamma\alpha' - \gamma'\alpha)} = \frac{C}{\gamma\gamma'(\alpha\beta' - \alpha'\beta)},$$

$$\text{or } \frac{A}{(b-c)\,l} = \frac{B}{(c-a)\,m} = \frac{C}{(a-b)\,n};$$

hence, the equations of the bisecting lines may be written,

$$\frac{l}{x}(b-c) + \frac{m}{y}(c-a) + \frac{n}{z}(a-b) = 0,$$

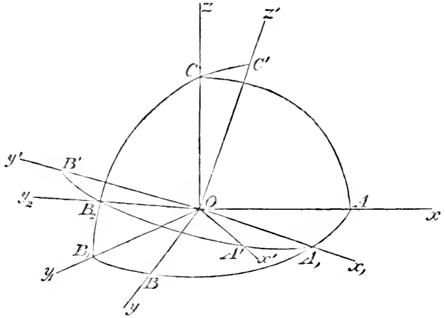
$$\text{and } lx + my + nz = 0.$$

146. *Euler's formulæ for transforming from one system of rectangular coordinates to another having the same origin.*

There being in the formulæ already obtained for this purpose nine constants connected by six invariable relations, it must be possible to obtain formulæ to effect this transformation which shall involve only three constants. The three chosen by Euler for this purpose are (1) the angle which the intersection of the planes of xy and $x'y'$ makes with the axis of x, (2) the angle

made by the same straight line with the axis of x', (3) the angle between the planes of xy and $x'y'$.

Let Ox, Oy, Oz be the original; Ox', Oy', Oz' the transformed axes of coordinates; Ox_1 the intersection of the planes of xy, $x'y'$; $xOx_1 = \phi$, $x'Ox_1 = \psi$, $zOz' = \theta$, which is the same as the angle between the planes of xy, $x'y'$.



The transformations may be effected by successive transformations, each in one plane—

(1) through an angle ϕ, in the plane of xy, from Ox, Oy to Ox_1, Oy_1;

(2) through an angle θ, in the plane of y_1z, from Oy_1, Oz to Oy_2, Oz';

(3) through an angle ψ, in the plane of y_2x_1, from Ox_1, Oy_2 to Ox', Oy'.

The formulæ for these transformations are, using the same suffix for any one of the coordinates as for the corresponding axis,

$$\left.\begin{aligned} x &= x_1\cos\phi - y_1\sin\phi, \\ y &= x_1\sin\phi + y_1\cos\phi, \end{aligned}\right\}$$

$$\left.\begin{aligned} y_1 &= y_2\cos\theta - z'\sin\theta, \\ z &= y_2\sin\theta + z'\cos\theta, \end{aligned}\right\}$$

$$\left.\begin{aligned} x_1 &= x'\cos\psi - y'\sin\psi, \\ y_2 &= x'\sin\psi + y'\cos\psi, \end{aligned}\right\}$$

from which we obtain, by successive substitutions,

$$x = x'(\cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta)$$
$$\qquad - y'(\cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta) + z'\sin\phi\sin\theta,$$
$$y = x'(\sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta)$$
$$\qquad - y'(\sin\phi\sin\psi - \cos\phi\cos\psi\cos\theta) - z'\cos\phi\sin\theta,$$
$$z = x'\sin\psi\sin\theta + y'\cos\psi\sin\theta + z'\cos\theta.$$

These formulæ might be established without successive transformation by Spherical Trigonometry, but this is left for the exercise of the student.

147. These formulæ are too complicated and unsymmetrical to be generally employed. A modification of them, however, may be useful in determining the nature of any proposed plane section of a surface. We may in that case, by using the first two transformations, make the plane of x_1y_2 coincide with the proposed plane section, and then, making $z'=0$, obtain the equation of the section in that plane.

The results may be at once derived from the final equations of the last article by making $\psi=0$, $z'=0$, or directly by geometrical considerations, and we have the formulæ

$$x = x'\cos\phi - y'\cos\theta\sin\phi,$$
$$y = x'\sin\phi + y'\cos\theta\cos\phi,$$
$$z = y'\sin\theta,$$

by effecting these substitutions we may obtain the equation of the curve which is the intersection of a surface with a given plane.

If the equation of the plane be $lx+my+nz=0$, and the curve of intersection with the surface $f(x, y, z)=0$ be required, we shall have

$$\cos\theta = \frac{n}{\sqrt{(l^2+m^2+n^2)}}, \text{ and } \frac{\sin\phi}{-l} = \frac{\cos\phi}{m} = \frac{1}{\sqrt{(l^2+m^2)}},$$

and the equation of the curve of intersection will be

$$f(x\cos\phi - y\cos\theta\sin\phi,\ x\sin\phi + y\cos\theta\cos\phi,\ y\sin\theta) = 0.$$

148. As an example of this method, we will examine the position of a plane passing through the origin, when its intersection with the surface $ax^2 + by^2 + cz^2 = 1$ is a circle.

Let the intersection of the plane with the plane of xy make an angle ϕ with Ox, and let θ be its inclination to the plane of xy.

The equation of the section will be

$$a(x\cos\phi - y\cos\theta\sin\phi)^2 + b(x\sin\phi + y\cos\theta\cos\phi)^2 + cy^2\sin^2\theta = 1,$$

if this be a circle, the coefficient of $xy = 0$, and those of x^2 and y^2 will be equal and positive;

$$\therefore\ (a-b)\cos\theta\sin 2\phi = 0,$$

and $a\cos^2\phi + b\sin^2\phi = (a\sin^2\phi + b\cos^2\phi)\cos^2\theta + c\sin^2\theta > 0$,

we shall therefore obtain the following systems of solutions if a, b, c be unequal:

I. $\cos\theta = 0,\ a\cos^2\phi + b\sin^2\phi = c > 0,$

$$\text{or}\ \frac{\cos^2\phi}{b-c} = \frac{\sin^2\phi}{c-a} = \frac{1}{b-a}.$$

II. $\phi = 0,\ a = b\cos^2\theta + c\sin^2\theta > 0,$

$$\text{or}\ \frac{\cos^2\theta}{c-a} = \frac{\sin^2\theta}{a-b} = \frac{1}{c-b}.$$

III. $\phi = \frac{1}{2}\pi,\ b = a\cos^2\theta + c\sin^2\theta > 0,$

$$\text{or}\ \frac{\cos^2\theta}{c-b} = \frac{\sin^2\theta}{b-a} = \frac{1}{c-a}.$$

If a, b, c be of the same sign and in order of magnitude, III will be the only admissible solution, and the cutting plane must pass through the axis corresponding to the mean coefficient.

If a and b be positive, c negative, $b > a$, II will be the only solution.

If a be positive, b and c negative, there will be no plane circular section through the origin.

149. *Transformation from one system of coordinates to another having the same origin, both systems being oblique.*

Let Ox, Oy, Oz and Ox', Oy', Oz' be the two systems; On, On', On'' the normals respectively to yz, zx, and xy, and let nx

denote the angle nOx, and so for the others. Then the algebraical distance of a point whose coordinates in the two systems are respectively x, y, z and x', y', z' from the plane of yz, is $x \cos nx$, and is also

$$x' \cos nx' + y' \cos ny' + z' \cos nz'.$$

Whence $x \cos nx = x' \cos nx' + y' \cos ny' + z' \cos nz'$,

and similarly,

$$y \cos n'y = x' \cos n'x' + y' \cos n'y' + z' \cos n'z',$$

$$z \cos n''z = x' \cos n''x' + y' \cos n''y' + z' \cos n''z',$$

the required formulæ, involving in this form twelve constants, but, as they may be written in the form

$$x = a_1 x' + a_2 y' + a_3 z',$$
$$y = b_1 x' + b_2 y' + b_3 z',$$
$$z = c_1 x' + c_2 y' + c_3 z',$$

where $a_1 = \dfrac{\cos nx'}{\cos nx}$, and similarly for the others, we see that really only nine constants are involved, and these are connected by three equations on account of the angles between the original axes being fixed, so that again there are only six disposable constants.

150. *Transformation from any one system of axes to any other.*

If we wish in any of the above transformations of the directions of the axes also to remove the origin, we may first remove the origin to the point (f, g, h), retaining the directions of the axes. This will give

$$x = x_1 + f, \quad y = y_1 + g, \quad z = z_1 + h,$$

x_1, y_1, z_1 being the coordinates of a point (x, y, z) referred to the system of axes through the new origin parallel to the primary system. Now changing the direction by transformations of the form

$$x_1 = a_1 x' + a_2 y' + a_3 z', \text{ \&c.},$$

we see that the most general transformation possible is obtained by formulæ of the form

$$x = f + a_1x' + a_2y' + a_3z',$$
$$y = g + b_1x' + b_2y' + b_3z',$$
$$z = h + c_1x' + c_2y' + c_3z'.$$

151. *To shew that the degree of an equation cannot be changed by transformation of coordinates.*

We can now prove the important proposition, that the degree of an equation cannot be altered by any transformation of coordinates: the degree of an equation meaning the greatest number which can be obtained by adding the indices of the coordinates involved in any term. For let $Ax^py^qz^r$ be a term in an equation of the n^{th} degree, such that $p+q+r=n$: this will be a type of all the terms of the n^{th} degree involved in the equation, any one of which may be obtained by assigning to A, p, q, r suitable values. Now on any transformation this term becomes

$$A\,(f+a_1x'+a_2y'+a_3z')^p\,(g+b_1x'+b_2y'+b_3z')^q\,(h+c_1x'+c_2y'+c_3z')^r,$$

and no term in this product rises beyond the degree $p+q+r$ or n. Hence the degree of an equation cannot be raised by transformation of coordinates; nor can it be depressed, for if by any transformation the degree be depressed, then on re-transformation, the degree of the equation so depressed would be raised to its original value, which we have seen to be impossible.

152. *Relations between coefficients of a ternary quadric before and after transformation of coordinates.*

We notice here that in the case of quadric functions, relations between the coefficients in the original and transformed functions may be obtained without the use of the formulæ of transformation.

The method of obtaining these relations depends upon the consideration that, if a quadric be the product of two linear factors, it will still be so after any transformation of coordinates has been effected.

The square of the distance of any point (x, y, z) from the origin being $x^2+y^2+z^2$, if the axes be rectangular, this expression will be unaltered in form when a change is made from one set of rectangular axes to another having the same origin.

Let $u \equiv ax^2+by^2+cz^2+2a'yz+2b'zx+2c'xy$ be any ternary quadric, and let h be supposed so chosen that $h(x^2+y^2+z^2)-u$ shall be the product of two linear functions; the condition that this shall be the case is (Art. 88)

$$(h-a)(h-b)(h-c)-a'^2(h-a)-b'^2(h-b)-c'^2(h-c)-2a'b'c'=0,$$

shewing that there are generally three such values of h.

Suppose now that, on transformation to another system of rectangular axes, u becomes

$$v \equiv \alpha x^2+\beta y^2+\gamma z^2+2\alpha' yz+2\beta' zx+2\gamma' xy,$$

then $h(x^2+y^2+z^2)-v$ will for the same values of h be the product of two linear factors;

$$\therefore \ (h-\alpha)(h-\beta)(h-\gamma)$$
$$-\alpha'^2(h-\alpha)-\beta'^2(h-\beta)-\gamma'^2(h-\gamma)-2\alpha'\beta'\gamma'=0.$$

The two cubics being satisfied by the same values of h, we may equate the coefficients and obtain three relations

$$a+b+c=\alpha+\beta+\gamma,$$

$$bc+ca+ab-a'^2-b'^2-c'^2=\beta\gamma+\gamma\alpha+\alpha\beta-\alpha'^2-\beta'^2-\gamma'^2,$$

$$abc-aa'^2-bb'^2-cc'^2+2a'b'c'=\alpha\beta\gamma-\alpha\alpha'^2-\beta\beta'^2-\gamma\gamma'^2+2\alpha'\beta'\gamma',$$

these three functions of the coefficients, $a+b+c$, &c. are called invariants of the quadric, being equal to the same functions of the corresponding coefficients of the transformed quadric.

153. In the more general case of transformation from any system of axes to any other, the square of the distance of (x, y, z) from the origin being

$$x^2+y^2+z^2+2yz\cos\lambda+2zx\cos\mu+2xy\cos\nu,$$

it is easily seen that the two cubics which determine h will be of the form

$$(h-a)(h-b)(h-c)-(h\cos\lambda-a')^2(h-a)-\ldots$$
$$+2(h\cos\lambda-a')(h\cos\mu-b')(h\cos\nu-c')=0,$$

and the corresponding invariants are the ratios of the coefficients.

154. *To transform from rectangular to polar coordinates.*

In the cases in which polar coordinates are required to be used, we may first transform the axes so that the axis of z is parallel to the line from which θ is measured, and the plane of zx parallel to the plane from which ϕ is measured. If when referred to these axes the coordinates of the pole be f, g, h, the formulæ expressing the rectangular in terms of the polar coordinates will be

$$x = f + r\sin\theta\cos\phi, \quad y = g + r\sin\theta\sin\phi, \quad z = h + r\cos\theta.$$

155. *To transform from a four-plane to a three-plane coordinate system.*

This is immediately effected by the substitution of

$$p - lx - my - nz \text{ for } x,$$

and by similar substitutions for y, z, w.

If the three planes terminating in D be taken for the three-plane system, and l, m, n be the sines of the angles which the edges DA, DB, DC make with the planes DBC, DCA, DAB respectively, we shall have to write lx, my, nz for x, y, z, and $s_0\left(1 - \frac{lx}{p_0} - \frac{my}{q_0} - \frac{nz}{r_0}\right)$ for w to effect the transformation.

156. *To transform from one four-point coordinate system to another.*

If the equations of the fundamental points of the second system, referred to the first, be

$$\lambda p + \mu q + \nu r + \rho s = 0, \text{ \&c.},$$

and p, q, r, s; p', q', r', s', be the coordinates of any plane in the two systems,

$$p' = \frac{\lambda p + \mu q + \nu r + \rho s}{\lambda + \mu + \nu + \rho}, \text{ \&c. (Art. 112)}$$

from which equations the formulæ required for transformation can be deduced.

157. The method to be employed in the transformation, when the tangential equation of a surface is to be found, the

equation to which is given in four-plane or tetrahedral coordinates, and *vice versa*, we shall defer until we have considered the general conditions of tangency.

IX.

(1) If $l_1m_1n_1$, $l_2m_2n_2$, $l_3m_3n_3$ be the direction-cosines of a system of rectangular axes, and if $\frac{a}{l_1}+\frac{b}{m_1}+\frac{c}{n_1}=0$, and $\frac{a}{l_2}+\frac{b}{m_2}+\frac{c}{n_2}=0$, then will $\frac{a}{l_3}+\frac{b}{m_3}+\frac{c}{n_3}=0$, and $a:b:c::l_1l_2l_3:m_1m_2m_3:n_1n_2n_3$.

(2) If $al_1^2+bm_1^2+cn_1^2=0=al_2^2+bm_2^2+cn_2^2=al_3^2+bm_3^2+cn_3^2$, shew that

$$l_1^2-m_1^2:l_2^2-m_2^2:l_3^2-m_3^2::m_1^2-n_1^2:m_2^2-n_2^2:m_3^2-n_3^2,$$

and that $l_1(m_2n_3+m_3n_2)+l_2(m_3n_1+m_1n_3)+l_3(m_1n_2+m_2n_1)=0$.

(3) Transform the equation $yz+zx+xy=a^2$, referred to rectangular axes, to an equation referred to another system, one of which makes equal angles with the original axes.

(4) Shew that, by the same transformation as in the last problem, the equation $x^2+y^2+z^2+yz+zx+xy=a^2$ is reduced to the form

$$4x^2+y^2+z^2=2a^2.$$

(5) Prove both analytically and geometrically that, if the three straight lines, each of which is perpendicular to two of three other straight lines, be perpendicular to each other, the other three straight lines will be at right angles to each other.

(6) The straight lines bisecting the angles between the straight lines given by the equations

$$lx+my+nz=0,\quad ax^2+2bxy+cy^2=0,$$

lie in the two planes

$$x^2\{alm-b(n^2+l^2)\}+xy\{a(m^2+n^2)-c(n^2+l^2)\}-y^2\{clm-b(m^2+n^2)\}=0.$$

(7) The equations of the straight lines bisecting the angles between the straight lines given by the equations $lx+my+nz=0$, $ax^2+by^2+cz^2=0$, may be put into the form

$$\begin{aligned}&l^2x^2\{-l^2(b-c)+m^2(c-a)+n^2(a-b)\},\\ &+m^2y^2\{\ \ l^2(b-c)-m^2(c-a)+n^2(a-b)\},\\ &+n^2z^2\{\ \ l^2(b-c)+m^2(c-a)-n^2(a-b)\},\end{aligned}$$

and $lx+my+nz=0$.

(8) The straight lines bisecting the angles between the two lines given by the equations

$$lx + my + nz = 0, \quad ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 0,$$

lie on the cone

$$x^2 (c'n - b'm) + \ldots + \ldots + yz \{c'm - b'n + (c - b) l\} + \ldots + \ldots = 0.$$

(9) If $ax^2 + by^2 + cz^2$ become $\alpha\xi^2 + \beta\eta^2 + \gamma\zeta^2$ by any transformation of coordinates, the positive and negative coefficients will be in like number in the two expressions.

(10) Employ the method of Art. 152 to reduce the equation

$$x^2 + y^2 + \tfrac{1}{2}yz + zx = a^2,$$

to the form

$$x^2 + \tfrac{5}{4}y^2 - \tfrac{1}{4}z^2 = a^2.$$

(11) Assuming the formulæ for transforming from a system of coordinate axes inclined at angles α, β, γ, to another inclined at angles, α', β', γ', to be

$$x = l_1\xi + m_1\eta + n_1\zeta, \quad y = l_2\xi + m_2\eta + n_2\zeta, \quad z = l_3\xi + m_3\eta + n_3\zeta,$$

prove that $1 = l_1^2 + l_2^2 + l_3^2 + 2l_2l_3 \cos\alpha + 2l_3l_1 \cos\beta + 2l_1l_2 \cos\gamma$,

with similar equations in m and n; and that

$$\cos\alpha' = m_1n_1 + m_2n_2 + m_3n_3$$
$$+ (m_2n_3 + m_3n_2) \cos\alpha + (m_3n_1 + m_1n_3) \cos\beta + (m_1n_2 + m_2n_1) \cos\gamma,$$

with similar equations in n, l, and l, m.

(12) If there be two systems of rectangular coordinates, and θ_1, θ_2, θ_3 be the angles made by the axes of x', y', z', with that of z, and ϕ_1, ϕ_2, ϕ_3 the angles made by the planes of zx', zy', zz' with that of zx, then will

$$\cot^2\theta_1 \cos^2(\phi_2 - \phi_3) = \cot^2\theta_2 \cos^2(\phi_3 - \phi_1) = \cot^2\theta_3 \cos^2(\phi_1 - \phi_2)$$
$$= -\cos(\phi_2 - \phi_3)\cos(\phi_3 - \phi_1)\cos(\phi_1 - \phi_2).$$

(13) Shew, by transformation of four-point coordinates, that the centre of gravity of a tetrahedron is also the centre of gravity of the tetrahedron formed by joining the centres of gravity of the faces.

(14) Shew, by the same method, that the centre of gravity of the surface of a tetrahedron is the centre of the sphere inscribed in the tetrahedron formed by joining the centres of gravity of the faces.

CHAPTER X.

ON CERTAIN SURFACES OF THE SECOND DEGREE.

158. BEFORE proceeding to discuss the general equation of the second degree, we think it advisable for the student to render himself familiar with some of the properties of the surfaces which are represented by the general equation. We shall therefore introduce him to the equations of these surfaces in their simplest forms, in which the axes of coordinates being in the direction of lines symmetrically situated with regard to these surfaces, the nature and properties of the surfaces will be more easily deduced. We hope that by following this plan we shall assist the student to understand more clearly the methods adopted in the general equations.

For this purpose we shall give geometrical definitions of the surfaces, and deduce equations from those definitions; and we shall shew *vice versâ* how from these equations the geometrical construction of those surfaces can be deduced.

The Sphere.

159. *To find the equation of a sphere.*

DEF. A sphere is the locus of a point, whose distance from a fixed point is constant. The fixed point is the centre and the constant distance the radius of the sphere.

Let (a, b, c) be the centre of the sphere, d the radius, (x, y, z) any point on the sphere;

$$\therefore\ (x-a)^2 + (y-b)^2 + (z-c)^2 = d^2.$$

This equation may be written in the general form

$$x^2 + y^2 + z^2 + Ax + By + Cz + D = 0,$$

the equation required.

160. Since the general equation of the sphere contains four arbitrary constants, the sphere may be made to satisfy four specific conditions.

It may be seen from geometrical considerations that, when four conditions are given, there may be only one sphere, or a limited number, or an infinite number of spheres, which satisfy the equations; at the same time the four conditions must be consistent with the nature of a sphere, and if this be the case, and the conditions be independent, there must be a limited number of spheres satisfying those conditions. For example, if four points be given through which a sphere is to pass, no three points can lie in one straight line; and if four points lie in one plane, they must also lie in a circle, otherwise no sphere could pass through them, and if such a condition be satisfied, an infinite number of spheres can be constructed, each of which contains the circle in which the four points lie; if the four points do not lie in a plane, so that the four conditions to be satisfied are independent, the sphere is completely determined.

Again, if four planes be given, each of which is to be touched by the sphere, no three of these must have one line of intersection, and the four cannot pass through one point, except under a condition, and in that case an infinite number of spheres can be drawn, touching the four planes. In other cases, eight spheres can be drawn satisfying the conditions.

Equation of a sphere under specific conditions.

161. *To find the equation of a sphere passing through a given point.*

Let (a, b, c) be the given point, and the equation of the sphere

$$x^2 + y^2 + z^2 + Ax + By + Cz + D = 0;$$

$$\therefore\ a^2 + b^2 + c^2 + Aa + Bb + Cc + D = 0;$$

$$\therefore\ x^2 + y^2 + z^2 + A(x - a) + B(y - b) + C(z - c) = a^2 + b^2 + c^2$$

is the equation required.

If the given point be the origin, the equation will become

$$x^2 + y^2 + z^2 + Ax + By + Cz = 0,$$

and the sphere may be made to satisfy three more conditions.

162. *To find the equation of a sphere which passes through two given points in the axis of z.*

Let c_1, c_2 be the distances of the given points from O; when $x=0$ and $y=0$, the equation must become $(z-c_1)(z-c_2)=0$; therefore the equation of the sphere is

$$x^2+y^2+(z-c_1)(z-c_2)+Ax+By=0.$$

If the sphere touch the axis of z, $c_1=c_2=\gamma$,

$$\therefore\ x^2+y^2+z^2+Ax+By-2\gamma z+\gamma^2=0.$$

163. *To find the equations of spheres which touch the three axes.*

Let the equation of the sphere be

$$x^2+y^2+z^2+Ax+By+Cz+D=0.$$

Since it touches the axis of x, let a be the distance from the origin; therefore when $y=0$ and $z=0$,

$$x^2+Ax+D=0,$$

the roots of which are each equal to $\pm a$;

$$\therefore\ A=\pm 2a \text{ and } D=a^2.$$

Similarly, y^2+By+a^2 is a complete square;

$$\therefore\ B=\pm 2a \text{ and } C=\pm 2a,$$

and the equations of the spheres which satisfy the given conditions are

$$x^2+y^2+z^2\pm 2ax\pm 2ay\pm 2az+a^2=0,$$

which are eight in number for any given value of a, corresponding to the different compartments of the coordinate planes.

164. *To find the equation of a sphere touching the plane of xy in a given point.*

Since the sphere meets the plane of xy only in the given point $(a, b, 0)$, when $z=0$, the equation must reduce to

$$(x-a)^2+(y-b)^2=0;$$

therefore the required equation of the sphere is

$$(x-a)^2+(y-b)^2+z^2+Cz=0.$$

165. *Interpretation of the expression*

$$(x-a)^2+(y-b)^2+(z-c)^2-d^2$$

in the equation of a sphere.

Let the equation of the sphere be

$$(x-a)^2+(y-b)^2+(z-c)^2-d^2=0,$$

and (x', y', z') be any point Q, C the centre of the sphere, and let a straight line through Q intersect the sphere in the points P, P', and have for its equations

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n}=r;$$

therefore at the points P and P'

$$(lr+x'-a)^2+(mr+y'-b)^2+(nr+z'-c)^2-d^2=0,$$

if r_1, r_2 be the roots of this equation,

$$r_1r_2=(x'-a)^2+(y'-b)^2+(z'-c)^2-d^2;$$

therefore the left side of the equation for any point (x', y', z') is

$$QP.QP', \text{ or } -QP.QP',$$

according as Q is without or within the sphere.

If Q be without the sphere it will be the square of a tangent drawn from Q to the sphere.

If Q be within, it will be the square of the radius of the small circle on the sphere whose centre is Q.

Def. The product of the segments QP, QP' is called the *power* of the sphere with respect to Q.

Cor. All tangents drawn from an external point to the sphere are equal.

On the Relations of two or more Spheres.

166. *To find the equation of the radical plane of two spheres.*

Def. The *radical plane* of two spheres is the locus of points, the powers of the two spheres with respect to which are equal.

Let the equations of the two spheres be

$$(x-a)^2+(y-b)^2+(z-c)^2-d^2\equiv u=0,$$

and $$(x-a')^2+(y-b')^2+(z-c')^2-d'^2\equiv u'=0.$$

The equation of the radical plane is therefore $u-u'=0$.

167. *To shew that the six radical planes of four spheres intersect in one point.*

Let $$u = 0,\ u' = 0,\ u'' = 0,\ u''' = 0$$

be the equations, in this form, of the four spheres.

The equations of the six radical planes are given by

$$u = u' = u'' = u''',$$

which intersect in one point determined by these equations.

DEF. The point of intersection of the six radical planes is called the *radical centre* of the four spheres.

Cylindrical Surfaces.

168. It has been seen that the locus of an equation $F(x, y) = 0$, which involves only two of the coordinates, is a cylindrical surface, of which the generating lines are parallel to the axis of the omitted coordinate. We shall now shew how to obtain the equation of certain cylindrical surfaces in which the generating lines are in a general direction.

169. *To find the equation of the cylindrical surface, whose generating lines are in a given direction and guiding curve an ellipse traced on the plane of xy.*

Let the equations of the guiding ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ and } z = 0 \qquad (1),$$

and (l, m, n) the direction of the generating lines.

Let the equations of any generating line be

$$\left.\begin{aligned} nx &= lz + \alpha \\ ny &= mz + \beta \end{aligned}\right\} \qquad (2).$$

At the point of intersection of the generating line with the guiding curve, the values of x, y, z in (1) and (2) being the same, we obtain as a general equation, after eliminating x, y, and z,

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = n^2, \qquad (3),$$

and since this is true for all positions of the generating line, eliminating α, β between (2) and (3),

$$\frac{(nx-lz)^2}{a^2}+\frac{(ny-mz)^2}{b^2}=n^2$$

is true for every point in the cylindrical surface, and is therefore its equation.

Conical Surfaces.

170. Def. A *conical surface* is a surface generated by a straight line which constantly passes through a given point, called the vertex, and is subject to some other condition.

171. *To find the equations of a conical surface, whose vertex is the origin, generated by a straight line, of which a guiding curve is an ellipse, whose centre is in the axis of z, and plane parallel to the plane of xy.*

Let the equations of the guiding ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ and } z=c, \qquad (1),$$

those of a generating line in any position,

$$x=\alpha z, \quad y=\beta z. \qquad (2).$$

Eliminating x, y, z, the coordinates of the point in which the generating line meets the guiding curve, satisfying (1) and (2) simultaneously,

$$\frac{\alpha^2c^2}{a^2}+\frac{\beta^2c^2}{b^2}=1. \qquad (3).$$

Since this equation is true for every position of the generating line, eliminating α, β from (2) and (3),

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2},$$

which is the required equation of the surface.

172. *To find the equation of the conical surface, whose vertex is any given point, and of which the section by the plane of xy is an ellipse whose axes are in the axes of x and y.*

Let the coordinates of the vertex be f, g, h, and the equations of the elliptic section be

$$z=0, \text{ and } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

and let the equations of any generating line be

$$x-f=\alpha(z-h), \text{ and } y-g=\beta(z-h) \qquad (1),$$

where this straight line meets the ellipse

$$x=f-\alpha h, \ \ y=g-\beta h;$$

$$\therefore \ \frac{(f-\alpha h)^2}{a^2}+\frac{(g-\beta h)^2}{b^2}=1 \qquad (2);$$

eliminating α and β from (1) and (2), we obtain for every point in the surface

$$\frac{(fz-hx)^2}{a^2}+\frac{(gz-hy)^2}{b^2}=(z-h)^2,$$

which is the equation required.

COR. 1. If l, m, n be the direction-cosines of any generating line

$$\frac{(fn-hl)^2}{a^2}+\frac{(gn-hm)^2}{b^2}=n^2.$$

COR. 2. The equation of an oblique circular cone is

$$(fz-hx)^2+(gz-hy)^2=a^2(z-h)^2,$$

if a be the radius of the circle in the plane of xy.

173. *To shew that there are two systems of circular sections of any oblique circular cone.*

When the circle which guides the motion of the generating line has equations $z=0$, $x^2+y^2=a^2$, the cone will be perfectly general, if we take the vertex in the plane of zx, and therefore f, 0, h for the coordinates of the vertex.

The equation of the cone will then be, as in the last article,

$$(fz-hx)^2+h^2y^2=a^2(z-h)^2,$$

this may be written in the form

$$h^2(x^2+y^2+z^2-a^2)=z\{2fhx-(f^2-h^2-a^2)z-2ha^2\}.$$

Hence, if the conical surface be cut by either of the planes

$$z=\alpha, \text{ or } 2fhx-(f^2-h^2-a^2)z-2ha^2=\beta,$$

the points of intersection will satisfy an equation of the form

$$x^2+y^2+z^2+Ax+By+C=0,$$

for all values of α and β, and the sections will therefore be plane sections of a sphere.

Therefore, there are two series of circular sections made by two systems of parallel planes.

174. The trace of the cone on the plane of zx, putting $y=0$, has for its equation

$$(fz-hx)^2-a^2(z-h)^2=0,$$

being the two generating lines which lie in that plane; and the equation of two planes in opposite systems, giving circular sections, is

$$(z-\alpha)\{2fhx-(f^2-h^2-a^2)z-2ha^2-\beta\}=0;$$

by adding these equations we obtain

$$h^2(x^2+z^2+Ax+Bz+C)=0,$$

which shews that the four points, in which these generating lines meet the two circular sections, lie in a circle; hence, the first system of planes makes the same angle with one generating line which the second system does with the other.

The Spheroids.

175. Def. A *spheroid* may be generated by the revolution of an ellipse about either axis.

If the axis of revolution be the minor axis, the surface is called an *Oblate Spheroid*, and if the major axis, a *Prolate Spheroid*.

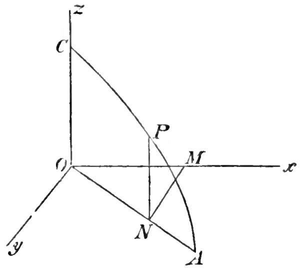
176. *To find the equation of a spheroid.*

Let the centre be taken as origin, the axis of revolution that of z, and let P be a point (x, y, z) in the ellipse CPA, which is the position of the revolving ellipse, when inclined at any angle to the plane of zx,

$$OM=x,\quad MN=y,\quad NP=z,\quad OA=a,\quad OC=c;$$

$$\therefore\ \frac{ON^2}{a^2}+\frac{NP^2}{c^2}=1;$$

$$\therefore\ \frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1.$$

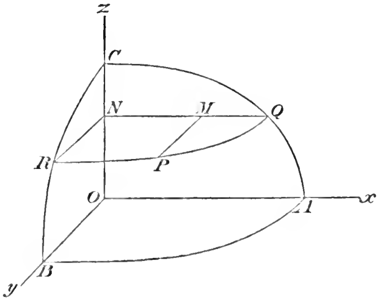


This is the equation of an oblate or prolate spheroid according as c is less or greater than a.

The Ellipsoid.

177. Def. An *ellipsoid* may be generated by the motion of a variable ellipse, which moves so that its plane is always parallel to a fixed plane, and which changes its form so that its vertices lie in two ellipses having a common axis traced on planes perpendicular to each other, and to the fixed plane.

178. *To find the equation of an ellipsoid.*



Let QRN be a variable ellipse in any position, Q, R being its vertices lying in two ellipses AC, BC, traced on perpendicular

planes, taken for those of zx and yz; the plane of xy, to which the variable ellipse is parallel, being the plane containing the semi-axes OA, OB.

Let a, c, and b, c, be the semi-axes of AC and BC, and (x, y, z) any point P in QR, PM perpendicular to QN.

$$\text{Then, } \frac{y^2}{RN^2} + \frac{x^2}{QN^2} = 1,$$

and, since Q is a point in the ellipse AC,

$$\frac{QN^2}{a^2} = 1 - \frac{z^2}{c^2},$$

$$\text{similarly, } \frac{RN^2}{b^2} = 1 - \frac{z^2}{c^2},$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

which is the equation required.

179. *To construct the surface whose equation is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Let the surface be cut by a plane whose equation is $z = \gamma$; the projection of the curve of intersection on the plane of xy has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{\gamma^2}{c^2},$$

therefore, the curve is an ellipse whose semi-axes α, β are given by the equations

$$\frac{\alpha^2}{a^2} = 1 - \frac{\gamma^2}{c^2} = \frac{\beta^2}{b^2};$$

hence, the vertices lie in the two ellipses which are the traces of the surface on the planes of zx and yz.

Also, since $\dfrac{\alpha}{a} = \dfrac{\beta}{b}$, the variable ellipse remains always similar to a given ellipse, which is the trace on the plane of xy.

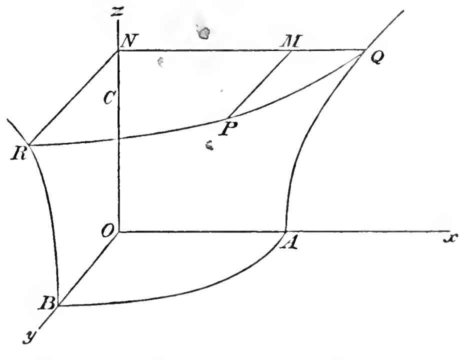
The surface may therefore be generated by the motion of a variable ellipse, whose plane, &c. (See Def.)

The Hyperboloid of one Sheet.

180. Def. The *hyperboloid of one sheet* may be generated by the motion of a variable ellipse, which moves so that its plane is always parallel to a fixed plane, and which changes its form so that its vertices always lie in two hyperbolas traced on planes perpendicular to each other and to the fixed plane, these hyperbolas having a common conjugate axis.

181. *To find the equation of an hyperboloid of one sheet.*

Let AQ, BR be the hyperbolas traced on the two perpendicular planes taken for the planes of zx, yz, OC their common semi-conjugate axis, being the direction of the axis of z.



Let QPR be the variable ellipse in any position, P any point (x, y, z) in it, QN, RN its semi-axes.

Draw PM perpendicular to QN.

$$\text{Then } MN = x, \quad PM = y, \quad ON = z,$$

$$\text{and } \frac{x^2}{QN^2} + \frac{y^2}{RN^2} = 1;$$

also, since Q, R are points in the hyperbolas,

$$\text{if } OA = a, \quad OB = b, \text{ and } OC = c,$$

$$\frac{QN^2}{a^2} = \frac{z^2}{c^2} + 1 = \frac{RN^2}{b^2};$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} + 1,$$

$$\text{or } \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

which is the equation of the hyperboloid of one sheet.

188. *To construct the surface which is the locus of the equation*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Let the surface be cut by a plane whose equation is $z = \gamma$, then the projection of the curve of intersection upon the plane of xy has for its equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{\gamma^2}{c^2},$$

which is the equation of an ellipse, whose semi-axes α, β are given by the equations

$$\frac{\alpha^2}{a^2} = 1 + \frac{\gamma^2}{c^2} = \frac{\beta^2}{b^2};$$

therefore the vertices of the ellipse lie respectively on the hyperbolas which are the traces of the surface on the planes of zx, yz.

Also, since $\dfrac{\alpha}{a} = \dfrac{\beta}{b}$, this ellipse is always similar to the ellipse which is the trace of the surface on the plane of xy.

Hence the locus may be generated by the motion of a variable ellipse which moves, &c. (See Def.)

183. The locus may also be generated by the motion of an hyperbola, for, if the surface be cut by a plane parallel to the plane of yz, whose equation is $x = \alpha$, the curve of intersection will be an hyperbola, the equation of whose projection on the plane of yz will be

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{\alpha^2}{a^2}.$$

If $\alpha < a$, the semi-axes β, γ will satisfy the equation

$$\frac{\beta^2}{b^2} = \frac{\gamma^2}{c^2} = 1 - \frac{\alpha^2}{a^2};$$

hence the extremities of the transverse axis 2β will lie on the ellipse, which is the trace on the plane of xy.

If $\alpha > a$, we shall have $\dfrac{\beta^2}{b^2} = \dfrac{\gamma^2}{c^2} = \dfrac{\alpha^2}{a^2} - 1$;

hence the extremities of the transverse axis 2γ will lie on the hyperbola, which is the trace on the plane of zx.

184. *To find the form of the surface at an infinite distance.*

If z be increased indefinitely,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}\left(1 + \frac{c^2}{z^2}\right) = \frac{z^2}{c^2} \text{ ultimately.}$$

Let this surface and the hyperboloid be cut by a straight line drawn parallel to Oz through a point $(x', y', 0)$, and let z_1, z_2 be the corresponding values of z,

$$\text{then } \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = \frac{z_1^{\ 2}}{c^2},$$

$$\text{and } \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = \frac{z_2^{\ 2}}{c^2} + 1,$$

$$\therefore \frac{z_1^{\ 2} - z_2^{\ 2}}{c^2} = 1, \text{ and } z_1 - z_2 = \frac{c^2}{z_1 + z_2};$$

if x', or y', or both, and therefore z_1 and z_2, be indefinitely increased, $z_1 - z_2$ will diminish indefinitely, and ultimately vanish;

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

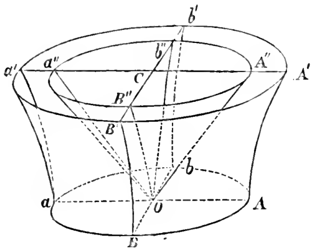
is the equation of an asymptotic surface, which lies further from the plane of xy than the hyperboloid.

This asymptotic surface is a cone, for, if it be cut by any plane whose equation is $\dfrac{x}{a} = \dfrac{z}{c}\cos\theta$, all the points of intersection will lie in the planes $\dfrac{y}{b} = \pm\dfrac{z}{c}\sin\theta$. The surface is therefore capable of being generated by a straight line which passes through the origin, and is guided by the ellipse whose equations are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ and } z = c.$$

The figure shews the position of the conical asymptote relative to the hyperboloid.

$ABab$ is the principal elliptic section, $A'B'a'b'$, $A''B''a''b''$ the sections of the hyperboloid and cone made by a plane parallel to that principal section, at a distance $OC=c$.



The Hyperboloid of two Sheets.

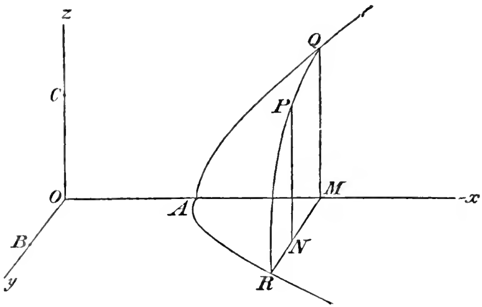
185. Def. The *hyperboloid of two sheets* may be generated by the motion of a variable ellipse, which moves so that its plane is always parallel to a fixed plane, and which changes its form, so that its vertices lie always on two hyperbolas having a common transverse axis, traced upon two planes perpendicular to each other and to the fixed plane.

186. *To find the equation of the hyperboloid of two sheets.*

Let AQ, AR be the hyperbolas traced on two perpendicular planes, taken for the planes of zx, xy, and having the common semi-transverse axis OA, and let QPR be the variable ellipse in any position, whose axes are QM, RM, parallel to the plane of yz.

Take P any point (x, y, z) in the ellipse, and draw PN perpendicular to RM, then $OM=x$, $MN=y$, and $NP=z$; therefore, since P is a point in the ellipse,

$$\frac{y^2}{RM^2}+\frac{z^2}{QM^2}=1\,;$$



and if a, c and a, b be the semi-axes of the two hyperbolas AQ, AR,

$$\frac{RM^2}{b^2} = \frac{x^2}{a^2} - 1 = \frac{QM^2}{c^2},$$

$$\therefore \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x^2}{a^2} - 1,$$

$$\text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

which is the equation of the hyperboloid of two sheets.

187. *To construct the locus of the surface whose equation is*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Let the surface be cut by a plane whose equation is $x = \alpha$; the equation of the projection on the plane of yz of the curve of intersection is

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{\alpha^2}{a^2} - 1,$$

which, if $\alpha > a$, is the equation of an ellipse whose semi-axes β, γ are given by the equations

$$\frac{\beta^2}{b^2} = \frac{\alpha^2}{a^2} - 1 = \frac{\gamma^2}{c^2},$$

therefore the vertices of the ellipse lie in two hyperbolas, whose equations are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ and } \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1,$$

which are the traces of the surface on the planes of xy, zx, having a common transverse axis in the line Ox; and since $\frac{\beta}{b} = \frac{\gamma}{c}$, this ellipse is always similar to a given ellipse, axes $2b$, $2c$.

Hence the locus may be constructed by the motion of a variable ellipse which, &c. (See Def.)

188. The locus may also be generated by the motion of a hyperbola; for, if the surface be cut by a plane parallel to the plane of xy, whose equation is $z = \gamma$, the curve of intersection will be an hyperbola, the equations of whose projection on the plane of xy will be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 + \frac{\gamma^2}{c^2},$$

which may be written $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$, whose transverse and conjugate semi-axes will satisfy the equations

$$\frac{\alpha^2}{a^2} = 1 + \frac{\gamma^2}{c^2} = \frac{\beta^2}{b^2}.$$

Hence the tranverse axis will have its extremities in the hyperbola, which is the trace on the plane of zx, and the hyperbolic section will be similar to the trace on the plane of xy.

189. *To find the form of the hyperboloid of two sheets at an infinite distance.*

If x be increased indefinitely, the equation $\frac{x^2}{a^2} = \frac{y^2}{b^2} + \frac{z^2}{c^2} + 1$ shews that y, or z, or both, will also be increased indefinitely, and the equation becomes

$$\frac{x^2}{a^2} = \left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\left(1 + \frac{1}{\frac{y^2}{b^2} + \frac{z^2}{c^2}}\right)$$

$$= \frac{y^2}{b^2} + \frac{z^2}{c^2} \text{ ultimately.}$$

Let the hyperboloid, and the surface represented by this equation, be cut by a straight line parallel to the axis of x, drawn through the point $(0, y', z')$, x_1, x_2, the corresponding values of x are given by the equations

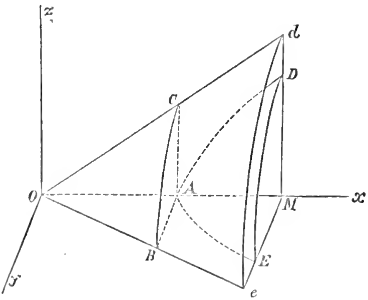
$$\frac{x_1^2}{a^2} = \frac{y'^2}{b^2} + \frac{z'^2}{c^2},$$

$$\text{and } \frac{x_2^2}{a^2} = \frac{y'^2}{b^2} + \frac{z'^2}{c^2} + 1;$$

$$\therefore \frac{x_2^2 - x_1^2}{a^2} = 1, \text{ and } x_2 - x_1 = \frac{a^2}{x_2 + x_1};$$

therefore $x_2 - x_1$ diminishes indefinitely, and ultimately vanishes as y', or z', or both, increase indefinitely; hence the hyperboloid of two sheets continually approximates to the form of the surface whose equation is $\frac{x^2}{a^2} = \frac{y^2}{b^2} + \frac{z^2}{c^2}$, which is therefore called an asymptotic surface.

Also, if this surface be cut by a plane whose equation is $\frac{y}{b} = \frac{x}{a}\cos\theta$, all the points of intersection will lie in the two planes $\frac{z}{c} = \pm\frac{x}{a}\sin\theta$; and the surface can therefore be generated by straight lines drawn through the origin, which intersect the ellipse, whose equations are $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $x = a$.



This asymptotic surface is therefore a cone on an elliptic base, and lies nearer to the plane of yz than the hyperboloid, since $x_1^2 < x_2^2$.

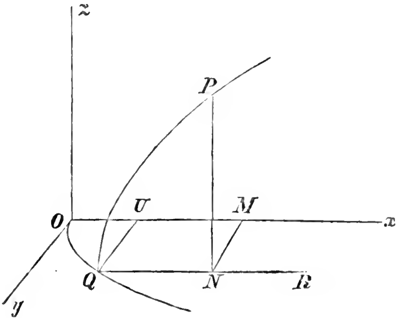
Its position relative to the hyperboloid is shewn in the figure, in which BC is the section made by a plane parallel to yz through the extremity of the transverse axis, and DE, de are sections of the hyperboloid and conical asymptote, made by a plane parallel to yz.

The Elliptic Paraboloid.

190. Def. The *elliptic paraboloid* may be generated by the motion of a parabola, whose vertex lies in a parabola traced upon a fixed plane, to which its plane is always perpendicular, the axes of the two parabolas being parallel, and the concavities turned in the same direction.

191. *To find the equation of the elliptic paraboloid.*

Let xOy be the plane on which the fixed parabola OQ is traced, Ox the axis of OQ; QR the axis of the moveable parabola QP, P any point (x, y, z) in the parabola.



Draw PN perpendicular to QR, and QU, NM to Ox, then

since P is a point in QP, if l, l' be the latera recta of OQ and QP,

$$PN^2 = l'.QN, \text{ and } QU^2 = l.OU;$$

$$\therefore \frac{y^2}{l} + \frac{z^2}{l'} = OU + QN = OM = x,$$

which is the equation of the elliptic paraboloid.

192. *To construct the locus of the equation*

$$\frac{y^2}{l} + \frac{z^2}{l'} = x.$$

Let the locus be cut by a plane, whose equation is $y = \beta$, the projection of the curve of intersection upon the plane of zx has for its equation

$$z^2 = l'\left(x - \frac{\beta^2}{l}\right),$$

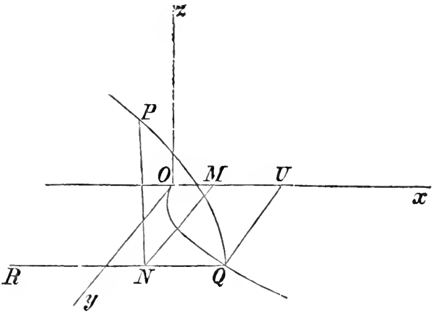
which represents a parabola whose axis is parallel to the axis of x, the coordinates of whose vertex are $\frac{\beta^2}{l}$, β, 0; therefore the vertex of the parabolic section lies in the parabola whose equation is $y^2 = lx$, which is the trace on the plane of xy; therefore the locus may be constructed by the motion of a parabola, whose vertex, &c. (See Def.)

The Hyperbolic Paraboloid.

193. Def. The *hyperbolic paraboloid* may be generated by the motion of a parabola, whose vertex lies in a parabola traced upon a fixed plane, to which its plane is perpendicular, the axes of the two parabolas being parallel, and the concavities turned in opposite directions.

194. *To find the equation of the hyperbolic paraboloid.*

Let xOy be the plane upon which the fixed parabola is drawn, Ox the direction of the axis of the parabola; let QR be the axis of the moveable parabola QP, parallel to Ox, measured in the direction contrary to Ox.



Draw PN perpendicular to QR, and QU, NM to Ox; then, if P be any point (x, y, z) in QP, $OM=x$, $MN=y$, and $NP=z$.

Let l, l' be the latera recta of OQ, QP;

$$\text{therefore, } PN^2 = l'.QN, \text{ and } QU^2 = l.OU,$$

$$\text{and } \frac{QU^2}{l} - \frac{PN^2}{l'} = OU - QN = OM;$$

$$\therefore \frac{y^2}{l} - \frac{z^2}{l'} = x,$$

which is the equation of the hyperbolic paraboloid.

195. *To construct the locus of the equation*

$$\frac{y^2}{l} - \frac{z^2}{l'} = x.$$

Let the locus of the equation be cut by the plane, whose equation is $y=\beta$; the projection of the curve of intersection upon the plane of zx has for its equation

$$z^2 = l'\left(\frac{\beta^2}{l} - x\right),$$

which represents a parabola, whose axis is measured in the direction contrary to Ox, and the algebraical distance of whose vertex from the plane yOz is $\frac{\beta^2}{l}$; therefore the section by the

plane $y=\beta$ is a parabola, whose latus rectum is l' and the co-ordinates of whose vertex are $\frac{\beta^2}{l}$, β, 0; or, the vertex lies in a parabola traced upon the plane of xy, whose equation is $y^2=lx$.

Hence the locus may be generated by the motion of a parabola, whose vertex, &c. (See Def.)

196. The locus may also be generated by the motion of an hyperbola; for if it be cut by a plane parallel to that of yz on the positive side, whose equation is $x=\alpha$, the equation of the projection of the curve of intersection on the plane of yz will be $\frac{y^2}{l}-\frac{z^2}{l'}=\alpha$, whose tranverse and conjugate semi-axes, β, γ, will satisfy the equations $\beta^2=l\alpha$ and $\gamma^2=-l'\alpha$, the extremities of the transverse axis will lie in the trace on the plane of xy, and the conjugate axis will be equal to the double ordinate of the trace on the plane of zx corresponding to $x=-\alpha$.

If it be cut by a plane parallel to yz on the negative side, the section will be an hyperbola whose transverse axis will be in the direction of Oz.

If $\alpha=0$, the hyperbolas will degenerate into two straight lines, which is the intermediate form in the transition.

197. *To find the form of the hyperbolic paraboloid at an infinite distance.*

If y and z be indefinitely increased while $x:z$ remains finite,

$$\frac{y^2}{l}=\frac{z^2}{l'}\left(1+\frac{l'x}{z^2}\right)=\frac{z^2}{l'} \text{ ultimately};$$

$$\therefore \frac{y}{\sqrt{l}}=\pm\frac{z}{\sqrt{l'}},$$

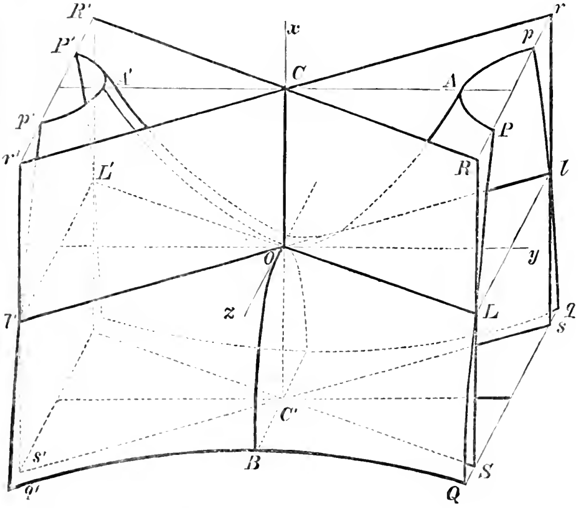
and if these planes and the hyperbolic paraboloid be cut by a straight line parallel to Oy, drawn through a point $(x', 0, z')$, y_1, y_2 the corresponding values of y will be given by the equations

$$\frac{y_1^{\ 2}}{l}=\frac{z'^2}{l'}, \text{ and } \frac{y_2^{\ 2}}{l}-\frac{z'^2}{l'}=x';$$

$$\therefore \frac{y_2^{\ 2}-y_1^{\ 2}}{l}=x', \text{ or } y_2-y_1=\frac{lx'}{y_2+y_1}.$$

Therefore, if x' remain finite or small compared with y_1 or y_2, $y_2 - y_1$ will diminish as z' increases and will ultimately vanish; and the two planes, whose equations are $\frac{y}{\sqrt{l}} = \pm \frac{z}{\sqrt{l'}}$, will give the form of the surface at an infinite distance for finite values of x, or for values of x which are small compared with y or z.

These planes will not form an asymptotic surface, except for points at which x vanishes compared with y or z, since $y_2 - y_1$ will not ultimately vanish in any other case, and similarly for $z_2 - z_1$.



The figure is intended to shew the position of the asymptotic planes with reference to the hyperbolic paraboloid.

Ox is parallel to the axis of the generating parabola, of which OB is one position in the plane of zx.

PAp, $P'A'p'$ are opposite branches of a hyperbolic section perpendicular to Ox, the asymptotes of which RCR', rCr' are sections of the asymptotic surface, AA' the transverse axis being parallel to Oy.

LL', ll' are the traces on the plane of yz of both the paraboloid and its asymptotic surface.

QBq' is a branch of a hyberbolic section on the negative side of Ox, the two asymptotes of which $SC'S'$, $sC's'$ are sections of the asymptotic surface, and the transverse axis BC' is parallel to Oz.

198. *To shew that the elliptic and hyperbolic paraboloid are particular cases of the ellipsoid, and the hyperboloid respectively.*

$$\text{Let } \frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1$$

be the equation of an ellipsoid or hyperboloid, and let the origin be removed to the point $(-a, 0, 0)$.

The transformed equation is

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = \frac{2x}{a}.$$

Let a, b, c become infinite, while $\frac{b^2}{a}$, $\frac{c^2}{a}$ remain finite quantities, and denote these by l and l'.

The equation may then be written

$$\frac{x^2}{a} \pm \frac{y^2}{l} \pm \frac{z^2}{l'} = 2x,$$

which has for its limit, when a becomes infinite,

$$\pm \frac{y^2}{l} \pm \frac{z^2}{l'} = 2x,$$

which is the equation of an elliptic or hyperbolic paraboloid. The assumption that $\frac{b^2}{a}$ and $\frac{c^2}{a}$ remain finite is the same thing as assuming that the latera recta of the traces on the planes xy, zx, respectively, remain finite when the axes become infinite, and the corresponding ellipses or hyperbolas become parabolas.

It is obvious from the above, that the elliptic paraboloid is a limiting case of either the ellipsoid or the hyperboloid of two sheets, and the hyperbolic paraboloid of the hyperboloid of one sheet.

199. The surfaces of the second degree, which we have been discussing, have equations of the two forms,

$$Ax^2 + By^2 + Cz^2 = D, \qquad (1)$$

$$\text{and } By^2 + Cz^2 = Ax; \qquad (2)$$

and it will be shewn in a succeeding chapter that the equations of all surfaces of the second degree may by transformation of coordinates be reduced to one of these two forms.

The first form of equation includes all surfaces which have a centre at a *finite* distance, and the second those which have a centre at an *infinite* distance.

In the equation (1), if $-x$, $-y$, $-z$ be written respectively for x, y, z, the equation will not be altered; therefore if (x, y, z) be a point in the surface, $(-x, -y, -z)$ also will be a point in it, so that if POP' be any chord through the origin O, the chord will be bisected in O, and O will be a centre of the surface.

Also, for any values of y and z, the values of x are equal and of opposite signs, therefore the plane of yz bisects the chords which are drawn perpendicular to it; and a plane which bisects the chords drawn perpendicular to it is called a *principal plane* of the surface.

Hence the planes xy, yz and zx are principal planes of the surface.

It is evident that the planes of zx, xy are principal planes of the surfaces whose equations are of the form (2).

The sections made by the principal planes are called *principal sections.*

That the surface represented by (2) may be considered to have a centre at an infinite distance may be shewn by considering this equation as the limiting form of (1) when the origin is transferred to a point $(-\alpha, 0, 0)$, α being determined by the equation $A\alpha^2 = D$. The equation (1) will then assume the form

$$Ax^2 + By^2 + Cz^2 = 2A\alpha x,$$

and this surface has a centre on the axis of x, at distance α from the origin.

Now, if we suppose A to vanish, while $A\alpha$ remains finite, an equation of the form (2) is the result. But to satisfy these

conditions α must be infinitely great; hence a surface represented by (2) has a centre at an infinite distance on the axis of x, and also a third principal section, parallel to the plane of yz, at an infinite distance.

200. Considering the peculiar importance of the properties of surfaces of the second degree, and their frequent occurrence in the solution of problems, and the establishment of theorems, in all departments of physical science, we have adopted a special term derived from the term *Conic*, invented by Salmon for the locus of the equation of the second degree in Plane Geometry.

DEF. The locus of the general equation of the second degree is called a *Conicoid*.*

X.

(1) A straight line is drawn through a fixed point O, meeting a fixed plane in Q, and in this straight line is taken a point P such that $OP.OQ$ is equal to a given quantity; shew that P lies on a sphere passing through O, whose centre lies on the perpendicular from O upon the plane.

(2) Investigate the equation of a sphere conceived to be generated by the motion of a variable circle, whose diameter is one of a system of parallel chords of a given circle, to which the plane of the variable circle is perpendicular.

(3) Construct the sphere whose polar equation is $r = a \sin\theta \cos\phi$.

(4) A straight line moves with three fixed points A, B, C in the three coordinate planes; shew that any other fixed point P of the straight line will lie on an ellipsoid whose semi-axes are equal to PA, PB, and PC.

(5) Find the locus of a point whose distance from a given point bears a constant ratio to its distance, (1) from a fixed plane, (2) from a fixed straight line.

(6) Find the locus of a point which is equidistant from two fixed lines which do not intersect.

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* The reasons for not adopting the term *Quadric*, which is employed by Salmon and approved of by many writers, are given in the Preface.

(7) The locus of a point, whose distance from a fixed plane is always equal to its distance from a fixed line, is a cone.

(8) Shew that the elliptic paraboloid may be generated by a variable ellipse, the extremities of whose axes lie on two parabolas having a common axis, and whose planes are at right angles to each other.

(9) Shew that an hyperboloid of one or two sheets degenerates into a right elliptic cone, when its axes become indefinitely small, and preserve a finite ratio to each other.

(10) Three straight lines, mutually at right angles, are drawn from the origin to meet the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, shew that, if their lengths be r_1, r_2, r_3,

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

(11) The curve traced out on the surface $\frac{y^2}{b} + \frac{z^2}{c} = x$ by the extremities of the latera recta of sections made by planes through the axis of x, lies on the cone $y^2 + z^2 = 4x^2$.

(12) The locus of the line of intersection of two planes at right angles to each other, each of which passes through one of two straight lines, inclined at an angle 2α, and whose shortest distance is $2c$, is a hyperboloid of one sheet, one of whose axes is $2c$, and the others are as $\cos\alpha : \sin\alpha$.

(13) The surface generated by a straight line, revolving about a fixed straight line, with which it is supposed rigidly connected, will be a cone, or a hyperboloid, according as the straight lines do or do not intersect.

(14) Find the equation of the locus of a line which always intersects two given lines, and is perpendicular to one of them. Interpret the result when the two given lines are at right angles to each other.

(15) The locus of the middle points of all straight lines passing through a fixed point and terminated by two fixed planes is a hyperbolic cylinder.

(16) Find the locus of straight lines which meet the two lines $x = a$, $y = 0$, and $x = -a$, $z = 0$, and touch the sphere $x^2 + y^2 + z^2 = c^2$; and shew that the locus reduces to two central conicoids when $c = a$.

(17) The ellipse, whose equations are $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and $z = mx$, rotates about the axis of z, prove that it always lies on the surface

$$x^2 + y^2 - (a^2 - b^2)\frac{z^2}{m^2a^2} = b^2.$$

(18) Prove that the cones on the elliptic base $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $z = 0$, whose vertices are on the hyperbola $\frac{x^2}{a^2 - b^2} - \frac{z^2}{b^2} = 1$, $y = 0$, are right circular.

(19) Of two equal circles, one is fixed and the other moves parallel to a given plane and intersects the former in two points; prove that the locus of the moving circle is two elliptic cylinders.

(20) If A, B, C be the extremities of the axes of an ellipsoid, and AC BC the sections containing the least axis, find the equations of the two cones whose vertices are A, B, and bases BC, AC respectively; shew that the cones have a common parabolic section, and if l be the latus-rectum of this parabola, and l_1, l_2 those of the sections AC, BC, then $\frac{1}{l^2} = \frac{1}{l_1^2} + \frac{1}{l_2^2}$.

(21) Find the locus of a point through which three straight lines can be drawn mutually at right angles, and passing through the perimeter of a curve whose equations are $z = 0$, and $ax^2 + by^2 = 1$.

(22) The trace of an ellipsoid on the plane of xy is AB; shew that a cone which has AB for a guiding curve will intersect the ellipsoid in another plane curve, and that this plane intersects the plane of AB in the polar with respect to AB of the projection of the vertex on that plane.

CHAPTER XI.

ON GENERATION BY STRAIGHT LINES.

201. In the preceding chapter we have shewn how certain surfaces of the second degree may be generated by the motion of ellipses, hyperbolas and parabolas. In the case of the cylinder and cone we have investigated the equations by supposing them to be generated by the motion of a straight line subject to certain conditions.

We shall in this chapter shew that the hyperboloid of one sheet, and the hyperbolic paraboloid, as well as the cone and cylinder, are capable of being generated by the motion of a straight line.

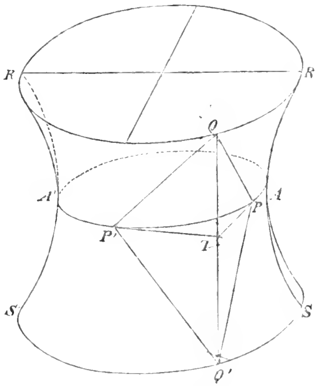
But, before giving the analytical representations of the mode of generation by straight lines, a general geometrical discussion may be found useful.

202. Since a surface of the second degree can be intersected by a straight line in two points only, unless it should turn out that the line lies entirely in the surface, as in the case of a cylinder, it follows that no straight line can intersect a plane section of the surface in more than two points, and that every plane section must therefore be a conic.

Now, if a plane be drawn containing a tangent to the principal elliptic section of the hyperboloid of one sheet and perpendicular to its plane, the curve of intersection with the surface will, in consequence of the flexure of the surface being in opposite directions, be a conic which crosses itself at the point of contact, and the only conic having this property is two intersecting straight lines.

Hence, through every point of the elliptic section two straight lines can be drawn which lie entirely in the surface, and by making the plane travel round the ellipse, such straight lines sweep round the whole surface, which can therefore be generated in two ways by the motion of a straight line.

203. If we take any two positions of the plane through tangents PT, $P'T$ to the principal elliptic section $APP'A'$,



their line of intersection, which will be perpendicular to the principal plane, will intersect the hyperboloid in two points only Q, Q', and the two pairs of generating lines will be PQ, PQ' and $P'Q$, $P'Q'$, since no straight line common to the plane PQQ' and the surface can meet QQ' except in Q or Q'.

Thus, the orthogonal projections of generating lines on this principal plane are tangents to the principal elliptic section; and similarly for the principal hyperbolic sections.

Also, through every point such as Q, two straight lines can be drawn which lie entirely in the surface; and it is evident that generating lines of the same system, such as PQ and $P'Q'$, do not intersect.

If a plane be drawn in any direction containing a generating line PQ, the conic of intersection must be two straight lines;

hence, the plane will contain another generating line, and these two generating lines will be of opposite systems, since they must intersect at a finite or infinite distance.

Since $P'Q$ is parallel to a generating line of the opposite system, drawn through the other extremity of the diameter through P', the same conical surface will be generated by lines drawn through the centre of the hyperboloid parallel to either system of generating lines.

204. *No straight line, which does not belong to one of the two systems of generating lines, lies on an hyperboloid.*

For, if possible, let a straight line (C) lie entirely on the hyperboloid, then since each system generates the whole hyperboloid, (C) must meet an infinite number of straight lines of each system; let two of these (A) and (B) of opposite systems intersect (C) in two different points, in which case a plane can be drawn through them intersecting the surface in three straight lines; but the section of a surface of the second order by a plane must be a curve of the second degree, therefore no such line as (C) can exist.

205. We leave it to the student to shew that a hyperbolic paraboloid may be generated in a similar way, and that the generating lines are all parallel to one or other of two fixed planes.

It will thus be seen that, since no three lines of a cone of the second degree can be parallel to the same plane, unless the cone split up into two planes, this forms a complete distinction between the two cases in which the generating lines of conicoids are real, *viz.* the hyperboloid of one sheet and the hyperbolic paraboloid.

206. *To find the surface generated by a straight line which meets three fixed non-intersecting straight lines.*

Let these fixed straight lines be (A), (B), (C); these lines obviously lie on the surface in question.

Now consider any plane through (A); it will meet (B) and (C) in points Q, R, and in these points only, and QR will meet (A) in some point P; so that PQR is the only straight

line of the system lying in the plane. Hence this plane meets the surface in two straight lines PQR and (A), which form a group of the second degree. But the section of a surface by a plane is a curve of the same degree as that of the surface. The surface in question is, therefore, of the second degree.

207. *To find in what cases a straight line can be drawn through a given point of a conicoid, so as to lie entirely in the surface.*

Let the equation of the conicoid, supposed central, be $ax^2 + by^2 + cz^2 = 1$, and let (f, g, h) be the given point, l, m, n the direction cosines of the straight line supposed to satisfy the condition; the coordinates of any other point at a distance r from (f, g, h) are $f + lr$, $g + mr$, $h + nr$; hence the equation

$$a(f + lr)^2 + b(g + mr)^2 + c(h + nr)^2 = 1,$$

must be satisfied for all values of r;

$$\therefore\ al^2 + bm^2 + cn^2 = 0, \qquad (1)$$

$$afl + bgm + chn = 0, \qquad (2)$$

$$\text{and } af^2 + bg^2 + ch^2 = 1. \qquad (3)$$

(1) shews that one or two of the quantities a, b, c must be negative; let c be negative; then since, by (1), (2), and (3),

$$(al^2 + bm^2)(af^2 + bg^2) - (afl + bgm)^2 = (1 - ch^2)(-cn^2) - c^2h^2n^2,$$

$$ab(gl - fm)^2 + cn^2 = 0; \qquad (4)$$

hence, unless a, b, or $c = 0$, which are cases of cylindrical surfaces, ab must be positive, and therefore both a and b will be positive, since all three cannot be negative.

Thus the central surface, on which a straight line can lie entirely, must be the hyperboloid of one sheet.

If the surface be non-central and its equation be $\dfrac{y^2}{b} + \dfrac{z^2}{c} = x$, the equation corresponding to (1) will be

$$\frac{m^2}{b} + \frac{n^2}{c} = 0,$$

which shews that the surface must be the hyperbolic paraboloid, since b and c must have opposite signs.

In this case, for every position of the point (f, g, h) there are two straight lines which lie entirely on the surface.

The hyperboloid of one sheet, and the hyperbolic paraboloid, can therefore be generated in two ways by the motion of a straight line.

208. The equations (2) and (4) give the directions of the two straight lines through (f, g, h),

$$\frac{l}{\pm\sqrt{\left(\frac{-cb}{a}\right)}\,g - cfh} = \frac{m}{\mp\sqrt{\left(\frac{-ca}{b}\right)}\,f - cgh} = \frac{n}{af^2 + bg^2}.$$

209. *To find the points of an hyperboloid of one sheet, corresponding to which the generating lines will be at right angles.*

By equations (1) and (2) of (Art. 207),

$$(afl + bgm)^2 + ch^2(al^2 + bm^2) = 0;$$

if l_1, m_1, n_1, and l_2, m_2, n_2, be the direction-cosines of the two generators

$$\frac{l_1 l_2}{m_1 m_2} = \frac{b(bg^2 + ch^2)}{a(af^2 + ch^2)};$$

$$\therefore \frac{al_1 l_2}{bg^2 + ch^2} = \frac{bm_1 m_2}{ch^2 + af^2} = \frac{cn_1 n_2}{af^2 + bg^2};$$

$$\therefore \frac{l_1 l_2}{\frac{1}{a} - f^2} = \frac{m_1 m_2}{\frac{1}{b} - g^2} = \frac{n_1 n_2}{\frac{1}{c} - h^2},$$

$$\text{and } l_1 l_2 + m_1 m_2 + n_1 n_2 = 0,$$

$$\therefore f^2 + g^2 + h^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

hence, the points lie on the intersection of the sphere

$$x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

with the hyperboloid.

210. The following method of dealing with generating lines, shews very clearly their relative positions.

Analysis of generating Lines.

211. *To find the generating lines of an hyperboloid of one sheet.*

The equation of the hyperboloid of one sheet is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1;$$

this equation may be written in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\cos\theta\pm\frac{z}{c}\sin\theta\right)^2+\left(\sin\theta\mp\frac{z}{c}\cos\theta\right)^2$$

for all values of θ;

$$\therefore\ \frac{x}{a}=\cos\theta\pm\frac{z}{c}\sin\theta \text{ and } \frac{y}{b}=\sin\theta\mp\frac{z}{c}\cos\theta, \qquad (1)$$

satisfy the equation; hence the two straight lines which, for a particular value of θ, have these for their equations, lie entirely in the surface.

By the variation of θ we obtain two systems of straight lines, which lie entirely in the surface, and either of these systems generates the hyperboloid. These equations may also be written in the form

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\pm\frac{z}{c},$$

from which equations it is manifest that straight lines drawn parallel to them through the centre will lie upon the asymptotic cone. Hence also, no three generators of the hyperboloid can be parallel to the same plane.

If $z=0$, $x=a\cos\theta$, and $y=b\sin\theta$; therefore θ is the eccentric angle of the point of intersection of the two straight lines (1) with the trace of the hyperboloid on the plane of xy.

212. Any point of the hyperboloid may be represented by the coordinates

$$a\cos\theta\sec\phi,\ b\sin\theta\sec\phi,\ c\tan\phi,$$

since these satisfy the equation for all values of θ and ϕ. The equations of the generating lines through this point may be readily found to be

$$\frac{x-a\cos\theta\sec\phi}{a\sin(\theta\pm\phi)}=\frac{y-b\sin\theta\sec\phi}{-b\cos(\theta\pm\phi)}=\frac{z-c\tan\phi}{\pm c};$$

which shew that they meet the principal elliptic section in points whose eccentric angles are $\theta \pm \phi$.

213. *The projections of the generating lines upon the principal planes are tangents to the traces on those planes.*

The equation of the trace on the plane of zx is

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1,$$

and that of the projection of a generating line on the same plane

$$\frac{x}{a} = \cos\theta \pm \frac{z}{c}\sin\theta,$$

and the points of intersection are given by the equation

$$\frac{z^2}{c^2} + 1 - \left(\cos\theta \pm \frac{z}{c}\sin\theta\right)^2 = 0,$$

$$\text{or } \frac{z^2}{c^2}\cos^2\theta \mp \frac{2z}{c}\cos\theta\sin\theta + \sin^2\theta = 0,$$

which, giving equal values of z, shews that the projection is a tangent to the trace upon the plane of zx.

Similarly, the projection on the plane of xy, and the trace on that plane, intersect in points given by the equations

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$\text{whence } \left(\frac{x}{a}\sin\theta - \frac{y}{b}\cos\theta\right)^2 = 0.$$

Hence the points of intersection coincide, or the projection is a tangent to the trace on the plane of xy.

214. *To shew that two generating lines of the same system do not intersect.*

The equations of two generating lines of the same system

$$\text{are } \frac{x}{a} = \cos\theta \pm \frac{z}{c}\sin\theta, \quad \frac{y}{b} = \sin\theta \mp \frac{z}{c}\cos\theta,$$

$$\text{and } \frac{x}{a} = \cos\theta' \pm \frac{z}{c}\sin\theta', \quad \frac{y}{b} = \sin\theta' \mp \frac{z}{c}\cos\theta';$$

if the two lines meet, we shall have at the points of intersection,

$$0 = \cos\theta - \cos\theta' \pm \frac{z}{c}(\sin\theta - \sin\theta'),$$

$$\text{and } 0 = \sin\theta - \sin\theta' \mp \frac{z}{c}(\cos\theta - \cos\theta');$$

and the condition of intersection will be

$$(\cos\theta - \cos\theta')^2 + (\sin\theta - \sin\theta')^2 = 0;$$

which cannot be satisfied unless $\theta = \theta'$.

Hence, generating lines of the same system do not intersect.

215. *To shew that generating lines of opposite systems must intersect.*

The equations of two generating lines of opposite systems

$$\text{are } \frac{x}{a} = \cos\theta \pm \frac{z}{c}\sin\theta, \quad \frac{y}{b} = \sin\theta \mp \frac{z}{c}\cos\theta;$$

$$\text{and } \frac{x}{a} = \cos\theta' \mp \frac{z}{c}\sin\theta', \quad \frac{y}{b} = \sin\theta' \pm \frac{z}{c}\cos\theta'.$$

If the two lines meet, we shall have at the points of intersection,

$$0 = \cos\theta - \cos\theta' \pm \frac{z}{c}(\sin\theta + \sin\theta'),$$

$$\text{and } 0 = \sin\theta - \sin\theta' \mp \frac{z}{c}(\cos\theta + \cos\theta'),$$

and the condition that they may intersect will be

$$\cos^2\theta - \cos^2\theta' + \sin^2\theta - \sin^2\theta' = 0,$$

which, being identically true, shews that any two generating lines of opposite systems intersect.

216. *To find the locus of the intersection of two generating lines of opposite systems, drawn through points in the principal elliptic section, whose eccentric angles differ by a constant angle.*

Let $\theta + \alpha$, and $\theta - \alpha$, be the eccentric angles of two points in the principal elliptic section, differing by a constant angle 2α.

The equations of the generating lines of opposite systems are

$$\frac{x}{a} = \cos(\theta + \alpha) \pm \frac{z}{c}\sin(\theta + \alpha), \quad \frac{y}{b} = \sin(\theta + \alpha) \mp \frac{z}{c}\cos(\theta + \alpha),$$

$$\text{and } \frac{x}{a} = \cos(\theta - \alpha) \mp \frac{z}{c}\sin(\theta - \alpha), \quad \frac{y}{b} = \sin(\theta - \alpha) \pm \frac{z}{c}\cos(\theta - \alpha).$$

At the points of intersection,

$$0 = \sin\theta \sin\alpha \mp \frac{z}{c} \sin\theta \cos\alpha, \quad \therefore \frac{z}{c} = \pm \tan\alpha.$$

$$\text{Also } \frac{x}{a} = \cos\theta \cos\alpha \pm \frac{z}{c} \cos\theta \sin\alpha = \cos\theta \sec\alpha,$$

$$\frac{y}{b} = \sin\theta \cos\alpha \pm \frac{z}{c} \sin\theta \sin\alpha = \sin\theta \sec\alpha;$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = \sec^2\alpha, \text{ and } \frac{z}{c} = \pm \tan\alpha.$$

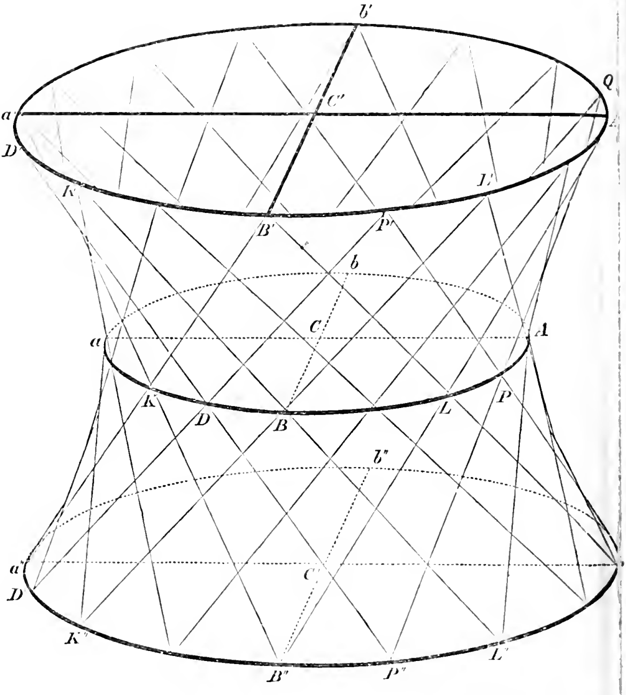
Therefore the locus required is the two elliptic sections, parallel to the plane of xy, which intersect the traces on the planes of zx, yz, at points whose eccentric angles are $\pm\alpha$.

217. The accompanying figure is meant to be a representation of the positions of sixteen generating lines of each system, corresponding to eccentric angles differing by $\frac{1}{8}\pi$. $ABab$ is the principal elliptic section, $A'B'a'b'$ and $A''B''a''b''$ are the parallel elliptic sections which intersect the conjugate axis of the hyperboloid at its extremities C', C'', the axes of which sections are in the ratio $\sqrt{2} : 1$ to the axes of the principal sections.

The generating lines through the extremities of the axes Aa, Bb intersect these two ellipses at points L', K', and L'', K'', whose eccentric angles are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, *i.e.* at the extremities of equi-conjugate diameters; and those through L, K, the extremities of equi-conjugate diameters of the principal elliptic section pass through the extremities of the axes of the two ellipses.

The two ellipses $A'B'a'$ and $A''B''a''$ are the loci of the intersections of opposite systems of generating lines drawn through the extremities of conjugate diameters of the principal elliptic section.

The figure serves to represent that the intersections of generating lines of opposite systems drawn through points in the principal elliptic section, whose eccentric angles differ by a constant angle, lie in an ellipse, the plane of which is parallel to the principal plane. As, for example, such pairs of generating lines as LB', $P'D$, and BL', PP'.



218. *To find the generating lines of a hyperbolic paraboloid.* The equation of the hyperbolic paraboloid,

$$\frac{y^2}{l} - \frac{z^2}{l'} = x,$$

is satisfied by the values of x, y, z for every point in the lines whose equations are

$$\frac{y}{\sqrt{l}} \mp \frac{z}{\sqrt{l'}} = \frac{\alpha}{\sqrt{l'}}, \qquad (1)$$

$$\text{and} \quad \frac{y}{\sqrt{l}} \pm \frac{z}{\sqrt{l'}} = \frac{\sqrt{l'}}{\alpha} x,$$

whatever be the value of α.

Therefore by giving α all values, we obtain two series of straight lines, all of which lie entirely in the surface; and these are the two systems of lines which are rectilinear generators of the paraboloid.

The equation (1) shews that in the two systems all the generators are parallel respectively to the two asymptotic planes, whose equations are

$$\frac{y}{\sqrt{l}} \mp \frac{z}{\sqrt{l'}} = 0.$$

219. *To shew that generating lines of a hyperbolic paraboloid of the same system do not intersect, and that those of opposite systems do intersect.*

Let the equations of two generating lines of the same system be

$$\frac{y}{\sqrt{l}} \mp \frac{z}{\sqrt{l'}} = \frac{\alpha}{\sqrt{l'}}, \quad \frac{y}{\sqrt{l}} \pm \frac{z}{\sqrt{l'}} = \frac{\sqrt{l'}}{\alpha} x,$$

$$\text{and } \frac{y}{\sqrt{l}} \mp \frac{z}{\sqrt{l'}} = \frac{\beta}{\sqrt{l'}}, \quad \frac{y}{\sqrt{l}} \pm \frac{z}{\sqrt{l'}} = \frac{\sqrt{l'}}{\beta} x.$$

If the two lines could intersect, these equations could be simultaneous, therefore $\alpha - \beta = 0$, which is impossible, since the two generating lines are distinct; hence they do not intersect.

Changing the order of the signs in the ambiguities in the second set of equations, we have the equations of a line in the system opposite to that of the first.

If then the straight lines intersect,

$$\frac{\alpha + \beta}{\sqrt{l'}} = \frac{2y}{\sqrt{l}} = \sqrt{l'}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) x,$$

$$\text{and } \frac{\alpha - \beta}{\sqrt{l'}} = \mp \frac{2z}{\sqrt{l'}} = \sqrt{l'}\left(\frac{1}{\beta} - \frac{1}{\alpha}\right) x,$$

and the consistency of these equations proves that two generating straight lines of opposite systems will always intersect.

220. *To shew that the projections of the generating lines on the principal planes are tangents to the principal sections.*

Since the equations of the generating lines are

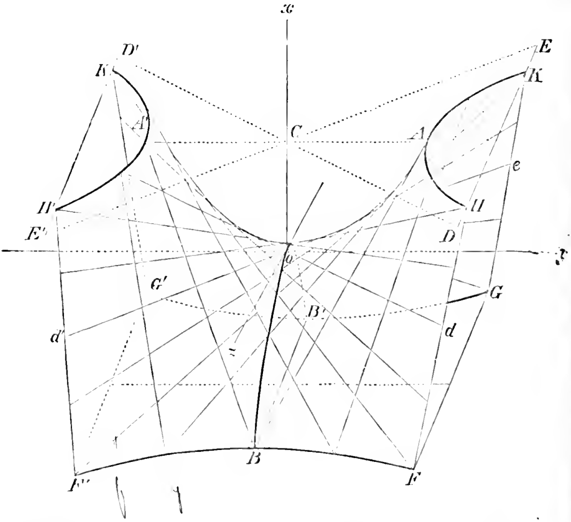
$$\frac{y}{\sqrt{l}} \mp \frac{z}{\sqrt{l'}} = \frac{\alpha}{\sqrt{l'}}, \quad \frac{y}{\sqrt{l}} \pm \frac{z}{\sqrt{l'}} = \frac{\sqrt{l'}}{\alpha} x,$$

the equations of their projections on the plane of zx, are

$$\pm \frac{2z}{\sqrt{l'}} = \frac{\sqrt{l'}}{a} x - \frac{a}{\sqrt{l'}},$$

which, being of the form $z = mx - \dfrac{l'}{4m}$, are the equations of tangents to the parabola $z^2 = -l'x$; similarly for the projections on the plane of xy.

221. The accompanying figure is intended to represent the manner in which the hyperbolic paraboloid is generated by straight lines.



HAK, $H'A'K'$ are portions of the branches of a hyperbolic section made by a plane parallel to that of yz, cutting Ox on the positive side; ECE', DCD' are the asymptotes of the section.

FBF'', $GB'G'$ are portions of the branches of a hyperbolic section parallel to yz on the negative side of Ox.

AOA' and BOB' are the traces on the planes of xy and zx.

The two sections are so chosen that the generating lines through B, an extremity of the transverse axis of one section, pass through A, A', the extremities of the transverse axis of the other.

dO, $d'Oe$ are the traces of the paraboloid on the plane yz, where the hyperbolic section degenerates into two straight lines.

XI.

(1) The equations of the generating lines of the surface

$$yz + zx + xy + a^2 = 0,$$

drawn through the point $\left(0,\ am,\ -\frac{a}{m}\right)$, are

$$x(1 \pm m) = am - y = \mp (mz + a).$$

(2) At any point where the planes $x + y + z = \pm a$ meet the surface $xy + yz + zx + a^2 = 0$, the two generating lines of the surface are at right angles to each other.

(3) The eccentric angles of the points in which the principal hyperbolic sections are met by any generating line are complementary, and that of the point in which it meets the principal elliptic section is equal to one of these.

(4) Prove that the points at a finite distance on a hyperbolic paraboloid, at which the generating lines are at right angles to each other, lie in a plane.

(5) Shew, by geometrical considerations, that the locus of intersection of two generating lines drawn through two points in the principal elliptic section of an hyperboloid of two sheets, whose eccentric angles differ by a constant quantity, is two ellipses parallel to the principal plane, at equal distances from it.

(6) Prove that, if any straight line intersect three straight lines which are all parallel to the same plane without intersecting each other, the intersecting straight line will in all positions be parallel to another fixed plane.

(7) Shew that there are two straight lines, and two only, which intersect four straight lines, no three of which are parallel to the same plane, and no two of which intersect.

(8) Find at what points of the principal elliptic section of an hyperboloid the generating lines can be at right angles, and shew that the diameter parallel to the tangent at that point is equal to the length of the imaginary axis.

(9) If four generating lines intersect so as to form a quadrilateral, whose angular points taken in order are $(\theta_1\phi_1)$, $(\theta_2\phi_2)$, $(\theta_3\phi_3)$, $(\theta_4\phi_4)$, (see Art. 212), prove that $\theta_1 + \theta_3 = \theta_2 + \theta_4$, and $\phi_1 + \phi_3 = \phi_2 + \phi_4$.

(10) A straight line moves so as to intersect the parabolas

$$y^2 = ax, \quad z = 0; \quad z^2 = -bx, \quad y = 0;$$

and to be always parallel to one of the planes $\frac{y}{\sqrt{a}} = \pm \frac{z}{\sqrt{b}}$; shew that its locus is the paraboloid $\frac{y^2}{a} - \frac{z^2}{b} = x$.

(11) The equation of the locus of a straight line constrained to move so as to intersect three straight lines, which do not intersect each other, and are not parallel to the same plane, is, when referred to axes parallel to the straight lines,

$$\frac{yz}{bc} + \frac{zx}{ca} + \frac{xy}{ab} - \frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0.$$

---

(12) The generating lines of the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, at any point where it is met by the cone $\frac{x^2}{a^6} + \frac{y^2}{b^6} - \frac{z^2}{c^6} = 0$, are both perpendicular to some other generating line.

If the generating lines be themselves at right angles, the point will lie also on the sphere $x^2 + y^2 + z^2 = a^2 + b^2 - c^2$. Shew that these conditions cannot coexist unless $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

(13) If three generating lines of the same system on an hyperboloid be mutually at right angles, the shortest distance between any two will lie on a generating line.

(14) If three generating lines of the same system, mutually at right angles, be made the edges of a rectangular parallelepiped, shew that the angular points of the parallelepiped which are not on the hyperboloid will lie on the surface $x^2 + y^2 + z^2 = a^2 + b^2 - c^2$, and on the surface whose equation would be obtained by eliminating h between the equations

$$\frac{x^2}{h^2 + a^2} + \frac{y^2}{h^2 + b^2} + \frac{z^2}{h^2 - c^2} + 1 = 0, \quad \frac{a^2x^2}{(h^2 + a^2)^2} + \frac{b^2y^2}{(h^2 + b^2)^2} - \frac{c^2z^2}{(h^2 - c^2)^2} = 1.$$

(15) If two planes be drawn, passing respectively through two generating lines of the same system at the extremities of the major axis of the principal elliptic section, and intersecting in a third generating line, the traces of these planes on either of two fixed planes will be at right angles to each other.

(16) If a ray of light be reflected between two plane mirrors, inclined at any finite angle, shew that all the reflected rays will lie on an hyperboloid of revolution; and find its position.

(17) The perpendiculars from the origin on the generating lines of the paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$ lie upon the cones $\left(\frac{x}{a} \pm \frac{y}{b}\right)(ax \pm by) + 2z^2 = 0$.

(18) The perpendiculars from the origin upon the generating lines of the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ lie upon the cone

$$\frac{a^2}{x^2}(b^2 + c^2)^2 + \frac{b^2}{y^2}(c^2 + a^2)^2 = \frac{c^2}{z^2}(a^2 - b^2)^2.$$

(19) The angle between two planes, each passing through the centre, and through one of the generating lines at any point of an hyperboloid, is given by the equation

$$\frac{2r \cot\phi}{abc} = \frac{1}{p^2} - \frac{1}{a^2} - \frac{1}{b^2} + \frac{1}{c^2},$$

r being the distance of the point, and p that of the plane containing the generating lines, from the centre of the hyperboloid.

(20) If θ be the acute angle between the perpendiculars from the centre on the generating lines of an hyperboloid which pass through the point $(a \cos\alpha, b \sin\alpha, 0)$, then will

$$\frac{c^2 (a^2 - b^2)^2}{\tan^2 \frac{1}{2}\theta} = \frac{a^2 (b^2 + c^2)^2}{\sin^2\alpha} + \frac{b^2 (c^2 + a^2)^2}{\cos^2\alpha}.$$

(21) If ϕ be the angle between the generating lines of the hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

which pass through a point at a distance r from the origin, and if p be the perpendicular from the origin upon the plane passing through them, shew that $2abc \cot\phi = p\,(r^2 - a^2 - b^2 + c^2)$.

(22) The tangent of the angle between the generating lines of the surface

$$\frac{x^2}{a} - \frac{y^2}{b} = z,$$

which pass through the point (f, g, h), is

$$\frac{\sqrt{\left(\frac{ab}{4} + \frac{bf^2}{a} + \frac{ag^2}{b}\right)}}{h + \frac{a - b}{4}}.$$

(23) Prove that if r be the distance of any point of the surface $yz + zx + xy + 2a^2 = 0$, from the origin, the angle between the two generating lines at that point will be

$$\cos^{-1}\frac{r^2 - 6a^2}{r^2 + a^2}.$$

(24) The angle between the generating lines through the point (xyz) of the hyperboloid $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$ is $\cos^{-1} \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}$, where λ_1, λ_2 are the roots of the equation

$$\frac{x^2}{a\,(a+\lambda)} + \frac{y^2}{b\,(b+\lambda)} + \frac{z^2}{c\,(c+\lambda)} = 0.$$

(25) Generating lines of the hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

are drawn through points in the plane of xy, whose eccentric angles are α, β; shew that their points of intersection are given by the equations

$$\frac{x}{a \cos \frac{\alpha+\beta}{2}} = \frac{y}{b \sin \frac{\alpha+\beta}{2}} = \frac{z}{\pm c \sin \frac{\alpha-\beta}{2}} = \frac{1}{\cos \frac{\alpha-\beta}{2}};$$

also that the shortest distance $\hat{c}$ between two of the same system is given by the equation

$$\frac{4 \sin^2 \frac{\alpha-\beta}{2}}{\hat{c}^2} = \frac{\sin^2 \frac{\alpha+\beta}{2}}{a^2} + \frac{\cos^2 \frac{\alpha+\beta}{2}}{b^2} + \frac{\cos^2 \frac{\alpha-\beta}{2}}{c^2}.$$

(26) The straight line which is orthogonal to each of two non-intersecting generators of the hyperboloid $x^2 + y^2 - z^2 = a^2$, becomes a generator of the opposite system when the two non-intersecting generators become consecutive.

(27) Shew that the shortest distances between generating lines of the same system drawn at the extremities of diameters of the principal elliptic section of the hyperboloid whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

lie on the surfaces, whose equations are

$$\frac{cxy}{x^2 + y^2} = \pm \frac{abz}{a^2 - b^2}.$$

(28) Shew that, if an eye observe the generating lines of an hyperboloid of one sheet, every generating line will appear to lie on another.

If the eye be placed upon the surface of the hyperboloid whose equation is $ax^2 + by^2 + cz^2 = 1$, prove that the points through which the generating lines appear to be perpendicular lie on a plane whose equation is

$$(a + b + c)\,(abx + bgy + chz - 1) = 2\,(a^2fx + b^2gy + c^2hz),$$

where (f, g, h) is the position of the eye.

CHAPTER XII.

SIMILAR SURFACES. PLANE SECTIONS OF CONICOIDS. CYCLIC SECTIONS.

222. We shall now consider the nature of the curves in which a plane intersects central and non-central conicoids, and we shall at present consider these surfaces as given by equations in the simplest form, such as have been discussed in the tenth chapter.

We shall examine the special cases in which the section made by a plane is circular, called a *cyclic* section, and the generation of the central conicoids and of the elliptic paraboloid by the motion of a variable circle, the plane of which is parallel to a given plane.

Similar Surfaces.

223. Def. Two surfaces are similar, say U and U', when for *any* point O determined with regard to U, and *any* two radii OP, OQ, another point O', and two radii $O'P'$, $O'Q'$ can be found for U', such that $\angle POQ = \angle P'O'Q'$, and

$$O'P' : O'Q' :: OP : OQ.$$

224. From the definition it follows that if OA, OB, OC be three arbitrary radii at right angles to one another in U, three radii $O'A'$, $O'B'$, $O'C'$ can be found also at right angles satisfying the above proportion, and if the direction cosines of radii OP and $O'P'$ referred to these as axes, in U and U' respectively, be equal, $OP : O'P' :: OA : O'A'$.

The surfaces will be similarly situated when the lines OA, OB, OC are parallel to $O'A'$, $O'B'$, $O'C'$, and in this case O may always be chosen so that O and O' coincide, in which

case the surfaces are said to be similarly situated with respect to O.

225. The analytical expression of this statement is that, if $f(x, y, z) = 0$ be the equation of any surface, that of any similar and similarly situated surface will be

$$f\{\lambda(x-\alpha), \lambda(y-\beta), \lambda(z-\gamma)\} = 0,$$

$$\text{where } OP = \lambda O'P'.$$

It is easily seen that the number of conditions, which the coefficients of the equations of two surfaces of the n^{th} degree must satisfy, is $\dfrac{(n+1)(n+2)(n+3)}{2.3} - 5$, in order that they may be similar and similarly situated.

Also, that the terms of the highest degree in the two equations must be the same, except for a constant factor.

Thus, in the case of the hyperboloids, they are similar if they have similar conical asymptotes.

It will be seen that, according to the definition, hyperboloids of one and two sheets may be similar, as

$$ax^2 + by^2 - cz^2 = 1$$

$$-ax^2 - by^2 + cz^2 = 1,$$

for imaginary radii of one drawn in the same direction as real radii of the other will be in the same ratio.

226. *Sections of the same conicoid by parallel planes are similar and similarly situated conics.*

Sections of similar and similarly situated conicoids by the same plane are similar and similarly situated conics.

These propositions are easily proved by transforming the axes of coordinates, so that the plane of xy is parallel to the cutting plane, when the projection of any section, found by making z constant, will be represented by an equation in x and y, for which the terms of the second degree will be the same.

Hence, we can deduce that a plane section of a hyperboloid is a hyperbola if the parallel plane through the centre intersects the conical asymptote in two of its generating lines.

227. It is of great importance to observe that, when two conicoids are similar and similarly situated, the condition, that the terms of the second degree are the same in each except for a constant factor, or, in geometrical language, that their real or imaginary asymptotes have their sheets parallel, may be stated as follows: "similar and similarly situated conicoids intersect the plane at infinity in the same real or imaginary conic."

A particular case of this is that "all spheres pass through the same imaginary circle at infinity."

228. *To determine the nature of the section of a conicoid made by any given plane.*

This may of course be done by the substitutions of Art. 147, but for surfaces of the second degree the plane sections will be curves of the second degree, so that simpler methods may advantageously be employed. If it be required only to discover the species of conic to which the section belongs, we may effect this immediately, taking any orthogonal projection of the curve of section, since an ellipse, parabola, or hyperbola, will be projected into a curve of the same species, though in general of different eccentricity. The only exception is when the plane of section is perpendicular to the plane of projection, but as no plane can be perpendicular to all the coordinate planes, there is at least one of the coordinate planes which may, in any proposed case, be taken as the plane of projection, and which will not be perpendicular to the plane of section.

As an example of this method, we may take the section of the paraboloid $by^2 + cz^2 = x$ made by the plane $lx + my + nz = 0$. The equation of the projection of the curve of section on the plane of yz is $l(by^2 + cz^2) + my + nz = 0$, which is always an ellipse, or always an hyperbola, according as b and c have like or unlike signs. If $l = 0$, the exceptional case above mentioned arises, and taking the projection on zx we have the equation

$$(n^2b + m^2c)\, z^2 = m^2x,$$

or the section is parabolic, unless $n^2b + m^2c = 0$, when it reduces to a straight line, the other straight line completing the curve of intersection being at an infinite distance. Hence, for the

paraboloids, all sections parallel to the axis of the principal sections are parabolas, and all other sections ellipses for the elliptic paraboloid, and hyperbolas for the hyperbolic paraboloid.

If, however, a more exact determination is required, it will be convenient to deal with the problem in the manner we propose.

Plane Sections.

229 *To find the locus of the centres of all sections of a central conicoid made by parallel planes.*

Let $ax^2 + by^2 + cz^2 = 1$ be the equation of the surface, and $lx + my + nz = p$ that of one of the parallel planes.

Any straight line drawn in this plane through the centre of the section will be bisected at that point.

Let (ξ, η, ζ) be the centre, r any radius of the section drawn in the direction (λ, μ, ν), therefore the values of r are given by

$$a(\xi + \lambda r)^2 + b(\eta + \mu r)^2 + c(\zeta + \nu r)^2 = 1, \qquad (1),$$

and, since the values of r are equal and of opposite signs,

$$a\xi\lambda + b\eta\mu + c\zeta\nu = 0, \qquad (2)$$

also, since the direction of r lies in the plane, we have

$$l\lambda + m\mu + n\nu = 0,$$

which equations being true for an infinite number of values of $\lambda : \mu : \nu$, we have

$$\frac{a\xi}{l} = \frac{b\eta}{m} = \frac{c\zeta}{n}, \qquad (3)$$

therefore the equations of the locus of centres of sections made by planes whose direction-cosines are l, m, n are

$$\frac{ax}{l} = \frac{by}{m} = \frac{cz}{n}.$$

230. The equation for determining r being

$$(a\lambda^2 + b\mu^2 + c\nu^2)\, r^2 = 1 - a\xi^2 - b\eta^2 - c\zeta^2, \qquad (4)$$

shews that the parallel plane sections are similar, for, if (λ', μ', ν') be the direction of another radius r', $r^2 : r'^2$ is independent of p, or constant for all the parallel sections, which are therefore similar and similarly situated.

231. *To find the position of the cutting plane when the curve of intersection becomes a point-ellipse or line-hyperbola.*

Since each member of (3) becomes

$$\frac{a\xi^2+b\eta^2+c\zeta^2}{l\xi+m\eta+n\zeta}=\frac{l\xi+m\eta+n\zeta}{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}, \text{ and } l\xi+m\eta+n\zeta=p,$$

equation (4) becomes $(a\lambda^2+b\mu^2+c\nu^2)\,r^2=1-\dfrac{p^2}{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$.

If ϖ be the value of p when the section becomes a point-ellipse or line-hyperbola, $r=0$ for any direction which makes $a\lambda^2+b\mu^2+c\nu^2$ finite; therefore $\varpi^2=\dfrac{l^2}{a}+\dfrac{m^2}{b}+\dfrac{n^2}{c}$.

The point-ellipse is when the values of λ, μ, ν given by $a\lambda^2+b\mu^2+c\nu^2=0$, and $l\lambda+m\mu+n\nu=0$ are impossible, and the line-hyperbola when they are real.

Now, it is not hard to shew that

$$(ma\lambda-lb\mu)^2+abc\nu^2\varpi^2=0,$$

hence, the section degenerates to a point-ellipse when abc is positive, or for an ellipsoid and hyperboloid of two sheets, and to a line-hyperbola when abc is negative, or for an hyperboloid of one sheet.

232. *To find the locus of the centres of all sections of an elliptic or hyperbolic paraboloid made by parallel planes.*

Let $by^2+cz^2=2x$ be the equation of a non-central conicoid, and let the equation of one of the parallel planes be

$$lx+my+nz=p,$$

then using the same notation as in the last article, we obtain the equation

$$b(\eta+\mu r)^2+c(\zeta+\nu r)^2=2(\xi+\lambda r), \qquad (1)$$

and deduce for an infinite number of values of $\lambda:\mu:\nu$

$$b\eta\mu+c\zeta\nu-\lambda=0, \qquad (2)$$

$$\text{and } m\mu+n\nu+l\lambda=0;$$

$$\text{therefore } \frac{b\eta}{m}=\frac{c\zeta}{n}=-\frac{1}{l}, \qquad (3)$$

thus, the locus of centres of sections made by planes whose direction-cosines are l, m, n is a straight line parallel to the axis of the paraboloid.

233. *To find the position of the plane for which the section is a point-ellipse or line-hyperbola.*

$$\text{Since } l\xi + m\eta + n\zeta = p,$$

$$l\xi = p + \left(\frac{m^2}{b} + \frac{n^2}{c}\right)\frac{1}{l} \text{ by (3)},$$

and the equation for determining r is

$$(b\mu^2 + c\nu^2)\, r^2 = 2\xi - b\eta^2 - c\zeta^2$$

$$= 2\xi - \left(\frac{m^2}{b} + \frac{n^2}{c}\right)\frac{1}{l^2}$$

$$= \frac{2p}{l} + \left(\frac{m^2}{b} + \frac{n^2}{c}\right)\frac{1}{l^2} = \frac{2}{l}(p - \varpi) \text{ suppose.}$$

The sections are ellipses, when r cannot be infinite, or when b and c are of the same sign; point-ellipses when $p = \varpi$.

They are hyperbolas where b and c are of opposite signs, and the directions of the asymptotes are given by $b\mu^2 + c\nu^2 = 0$, which shews that the asymptotes are parallel to the same two planes for all values of l, m, and n.

234. *To find the magnitude and direction of the axes of any plane central section of a central conicoid, and the area when the section is elliptic.*

The equations which connect the direction of any radius of the section by a plane, whose equation is $lx + my + nz = 0$, are, as in Art. 229,

$$(a\lambda^2 + b\mu^2 + c\nu^2)\, r^2 = 1 = \lambda^2 + \mu^2 + \nu^2,$$

$$\text{and } l\lambda + m\mu + n\nu = 0, \tag{1}$$

$$\therefore\; n^2\{(ar^2 - 1)\lambda^2 + (br^2 - 1)\mu^2\} + (cr^2 - 1)(l\lambda + m\mu)^2 = 0, \tag{2}$$

is an equation which, for a given length r, gives generally two values of $\lambda : \mu$; but if the given length be that of either semi-

axis, the two values of $\lambda : \mu$ are equal, the condition for which is

$$\{(ar^2-1)n^2+(cr^2-1)l^2\}\{(br^2-1)r^2+(cr^2-1)m^2\}=(cr^2-1)^2l^2m^2,$$

$$\therefore\ l^2(br^2-1)(cr^2-1)+\ldots=0, \qquad (3)$$

$$\text{or } \frac{l^2}{ar^2-1}+\frac{m^2}{br^2-1}+\frac{n^2}{cr^2-1}=0; \qquad (4)$$

this quadratic in r^2 gives the squares of the semi-axes of the section.

If 2α, 2β be the axes of the section,

$$\frac{1}{\alpha^2\beta^2}=l^2bc+m^2ca+n^2ab,$$

$$\text{and } \frac{1}{\alpha^2}+\frac{1}{\beta^2}=l^2(b+c)+m^2(c+a)+n^2(a+b).$$

When the section is elliptic its area

$$=\pi\alpha\beta=\frac{\pi}{\sqrt{(l^2bc+m^2ca+n^2ab)}}.$$

Again, the coefficient of λ^2 in (2) is $n^2(ar^2-1)+l^2(cr^2-1)$, which, by (3), is easily reduced to $-\dfrac{m^2(cr^2-1)(ar^2-1)}{br^2-1}$, hence the equation (2) becomes

$$\frac{ar^2-1}{br^2-1}m^2\lambda^2-2lm\lambda\mu+\frac{br^2-1}{ar^2-1}l^2\mu^2=0;$$

$$\therefore\ (ar^2-1)\frac{\lambda}{l}=(br^2-1)\frac{\mu}{m}=(cr^2-1)\frac{\nu}{n}, \qquad (5)$$

and, if we write α and β for r, these equations, with (1), will determine completely the directions of the two axes.

The equations (4) and (5) might have been formed by making r^2 a maximum or minimum, but we leave this to the student, the process adopted being more instructive.

235. *To find the direction of the plane section whose axes are of a given magnitude.*

The equation giving the axes in terms of the direction of the plane section is

$$\frac{l^2}{ar^2-1}+\frac{m^2}{br^2-1}+\frac{n^2}{cr^2-1}=0,$$

whence, if γ, δ be the reciprocals of the squares of the given axes,

$$l^2bc \quad + \quad m^2ca \quad + \quad n^2ab \quad = \gamma\delta,$$

$$l^2(b+c)+m^2(c+a)+n^2(a+b)=\gamma+\delta,$$

$$l^2 \quad + \quad m^2 \quad + \quad n^2 \quad = \quad 1;$$

multiplying the second and third equations by $-a$, a^2, and adding, we obtain, for the determination of l, m, and n,

$$l^2(a-b)(a-c)=(a-\gamma)(a-\delta);$$

similarly, $m^2(b-c)(b-a)=(b-\gamma)(b-\delta)$,

and $n^2(c-a)(c-b)=(c-\gamma)(c-\delta)$;

the second equation shews also that if a, b, c be in order of magnitude, b must be intermediate between γ and δ,

Hence, for a circular section, $\gamma=b=\delta$

$$\therefore m=0, \text{ and } \frac{l^2}{a-b}=\frac{1}{a-c}=\frac{n^2}{b-c}.$$

236. *To determine the nature of a central plane section of a central conicoid.*

The nature of the section may be determined from the discussion of the roots of the equation (3) obtained in Art. 234, viz.

$$(l^2bc+m^2ca+n^2ab)\,r^4-\{l^2(b+c)+m^2(c+a)+m^2(a+b)\}\,r^2+1=0, \quad (1)$$

observing that the discriminant

$$\{l^2(b+c)+m^2(c+a)+n^2(a+b)\}^2-4(l^2bc+m^2ca+n^2ab)$$

can be reduced to the form

$$\{l^2(b-c)+m^2(c-a)-n^2(a-b)\}^2+4l^2m^2(a-c)(b-c). \quad (2).$$

(1) For a hyperbolic section, the values of r^2 obtained from (1) must be of opposite signs; therefore $l^2bc+m^2ca+n^2ab$ is negative, in which case the values of r^2 must be real.

(2) For an elliptic section, the values of r^2, which by (2) are real if a, b, c be in order of magnitude, must be both positive;

$$\therefore \; l^2bc+m^2ca+n^2ab \text{ is positive.}$$

(3) For a rectangular hyperbolic section, the values of r^2 must be equal and of contrary signs;

$$\therefore \; l^2(b+c)+m^2(c+a)+n^2(a+b)=0,$$

since no real or finite values of a, b, c will, at the same time, admit of $l^2bc + m^2ca + n^2ab = 0$ as a consistent equation.

(4) For a circular section, the two values of r^2 are equal and of the same sign; therefore the expression (2) is zero, hence

$$m = 0, \text{ and } \frac{l^2}{a-b} = \frac{n^2}{b-c} = \frac{1}{a-c},$$

$$\text{or } l = 0, \text{ and } \frac{m^2}{a-b} = \frac{n^2}{c-a} = \frac{1}{c-b},$$

of which only one is possible.

(5) When $l^2bc + m^2ca + n^2ab = 0$, one of the roots is infinite, and the section becomes two parallel generating lines, the distance between which is $2r$, where r is given by

$$\{l^2(b+c) + m^2(c+a) + n^2(a+b)\} r^2 = 1.$$

From this analysis it appears that all sections of ellipsoids are ellipses, but that for the hyperboloids we may have ellipses, hyperbolas, or parallel straight lines.

237. We have in this discussion considered only central sections, but the nature of any section and the magnitude of the axes may be found at once by considering the similarity of parallel sections, the sections corresponding to parallel straight lines being parabolic; or, by writing $\dfrac{\varpi^2 r^2}{\varpi^2 - p^2}$ instead of r^2, the argument may be carried out in the same manner as above.

238. *To find the angle between the real or imaginary asymptotes of a plane section.*

From the equation $(a\lambda^2 + b\mu^2 + c\nu^2) r^2 = 1$,

$$\text{when } r = \infty, \; a\lambda^2 + b\mu^2 + c\nu^2 = 0,$$

$$\text{and } l\lambda + m\mu + n\nu = 0,$$

if ω be the angle between the asymptotes of the section, supposed hyperbolic, it may be shewn by the method of Art. 25 that

$$\cot\omega = \frac{l^2(b+c) + m^2(c+a) + n^2(a+b)}{2\sqrt{\{-(l^2bc + m^2ca + n^2ab)\}}},$$

or it may be obtained directly from the quadratic in r^2 (Art. 234), since $\tan^2\dfrac{\omega}{2} = \dfrac{-\beta^2}{\alpha^2}$, and therefore $\cot^2\omega = \dfrac{(\alpha^2+\beta^2)^2}{-4\alpha^2\beta^2}$.

This gives the condition that $\cot\omega$ will be real, infinite, or impossible as $l^2bc + m^2ca + n^2ab$ is negative, zero, or positive, thus determining the condition that the section may be hyperbolic, parabolic, or elliptic.

239. *To find the area of any elliptic section of a central conicoid made by a plane not passing through the centre.*

Let the equation of the plane be $lx + my + nz = p$; the area of a central elliptic section has been shewn to be

$$\frac{\pi}{\sqrt{(l^2bc + m^2ca + n^2ab)}},$$

and any radius vector of the section considered is given by

$$(a\lambda^2 + b\mu^2 + c\nu^2)\, r^2 = 1 - \frac{p^2}{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}},$$

hence, ϖ being the value of p when the section vanishes,

$$\varpi^2 = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c},$$

and, since $l^2bc + m^2ca + n^2ab$ is positive, ϖ is real only when abc is positive, or for the ellipsoid and hyperboloid of two sheets.

But if we take ϖ for the value of p when the section of the hyperboloid of two sheets, which is conjugate to that of one sheet, vanishes, since in this case a, b, c have their signs changed

$$\varpi^2 = -\frac{l^2}{a} - \frac{m^2}{b} - \frac{n^2}{c}.$$

Hence, if A be the area of the parallel central section of the ellipsoid and hyperboloid of one sheet, and of the hyperboloid conjugate to the hyperboloid of two sheets; and A' be the area of the section by the given plane, since they are in the duplicate ratio of homologous lines,

for the ellipsoid $A' = A\left(1 - \dfrac{p^2}{\varpi^2}\right)$,

for the hyperboloid of two sheets $A' = A\left(\dfrac{p^2}{\varpi^2} - 1\right)$,

for the hyperboloid of one sheet $A' = A\left(1 + \dfrac{p^2}{\varpi^2}\right)$.

If we take two conjugate hyperboloids $ax^2 + by^2 + cz^2 = \pm 1$, and the asymptotic cone to both, $ax^2 + by^2 + cz^2 = 0$, the area of the section of the latter may be found, from those of the former, by making a, b, c infinitely large, preserving their ratios. Hence if A_1, A_2, A_3 be the areas of the sections of the three surfaces made by any plane cutting them all in ellipses, and A that of the parallel central section of the hyperboloid of one sheet, we shall have

$$A_1 = A\left(1 + \frac{p^2}{\varpi^2}\right), \quad A_2 = A\left(\frac{p^2}{\varpi^2} - 1\right), \quad A_3 = A\,\frac{p^2}{\varpi^2},$$

whence $A_1 + A_2 = 2A_3$, or the section of the cone is an arithmetic mean between the sections of the two hyperboloids.

Also, if V be the volume of the cone cut off by a plane touching the hyperboloid of two sheets, we shall have

$$V = \tfrac{1}{3}A_3\varpi = \tfrac{1}{3}A\varpi.$$

Now $A = \dfrac{\pi}{\sqrt{(l^2bc + m^2ca + n^2ab)}}$, and $\varpi^2 = \dfrac{l^2}{a} + \dfrac{m^2}{b} + \dfrac{n^2}{c}$;

$$\therefore\ V = \tfrac{1}{3}\,\frac{\pi}{\sqrt{(abc)}},$$

which is constant for all positions of the cutting plane.

240. *To find the magnitude and direction of the axes of any plane section of an elliptic or hyperbolic paraboloid.*

The equation of the paraboloid being

$$by^2 + cz^2 = 2x,$$

and that of the cutting plane

$$lx + my + nz = p,$$

the equations connecting any central radius of the section with its direction (λ, μ, ν) is

$$(b\mu^2 + c\nu^2)\,r^2 = \frac{2}{l}\,(p - \varpi) = M \text{ suppose (Art. 233)}$$

$$= M(\lambda^2 + \mu^2 + \nu^2),$$

$$\text{and } l\lambda + m\mu + \nu n = 0;$$

$$\therefore\ l^2(b\mu^2 + c\nu^2)\,r^2 = M\{(m\mu + n\nu)^2 + l^2(\mu^2 + \nu^2)\}, \qquad (1)$$

and if r be the length of either of the semi-axes, this equation will give equal values of $\mu : \nu$,

$$\{l^2(br^2 - M) - m^2M\}\{l^2(cr^2 - M) - n^2M\} = M^2m^2n^2,$$

$$\text{or } l^2(br^2 - M)(cr^2 - M) - Mm^2(cr^2 - M) - Mn^2(br^2 - M) = 0,$$

$$\text{or } l^2bcr^4 - \{l^2(b+c) + m^2c + n^2b\}Mr^2 + M^2 = 0. \qquad (2)$$

This equation gives the magnitude of the axes 2α, 2β, and the area of the section when elliptic

$$= \pi\alpha\beta = \frac{\pi M}{l\sqrt{(bc)}} = \frac{2\pi}{l^2\sqrt{(bc)}}(p - \varpi).$$

The coefficient of μ^2 in equation (1), is

$$l^2(br^2 - M) - m^2M = Mn^2\frac{br^2 - M}{cr^2 - M},$$

and the equation becomes

$$\frac{br^2 - M}{cr^2 - M}\,.\,n^2\mu^2 - 2mn\mu\nu + \frac{cr^2 - M}{br^2 - M}\,m^2\nu^2 = 0\,;$$

$$\therefore\ (br^2 - M)\frac{\mu}{m} = (cr^2 - M)\frac{\nu}{n} = -M\frac{\lambda}{l}, \text{ by (2)},$$

which, writing α, β for r, completely determine the directions of the corresponding axes of the section.

241. *To determine the nature of any plane section of a paraboloid.*

Take the equation

$$l^2bcr^4 - \{l^2(b+c) + m^2c + n^2b\}Mr^2 + M^2 = 0,$$

and observe that the discriminant

$$\{l^2(b+c) + m^2c + n^2b\}^2 - 4l^2bc(l^2 + m^2 + n^2), \qquad (1)$$

is reducible to

$$\{l^2(b-c) - m^2c + n^2b\}^2 + 4m^2n^2bc. \qquad (2)$$

The two forms (1) and (2) of the discriminant shew that it is positive whether bc be positive or negative, so that the values of $r^2 : M$ are always real.

(1) For an elliptic section, bc is positive and M positive, therefore $p - \varpi$ has the same sign as l.

(2) For a hyperbolic section, bc is negative, since the values of r^2 must be of opposite signs.

(3) For a rectangular hyperbola, bc is negative, and

$$l^2(b+c) + m^2c + n^2b = 0.$$

(4) For a circular section, bc is positive, and by (2)

$$m = 0 \quad \text{and} \quad \frac{l^2}{b} = \frac{n^2}{c-b} = \frac{1}{c},$$

$$\text{or} \quad n = 0 \quad \text{and} \quad \frac{l^2}{c} = \frac{m^2}{b-c} = \frac{1}{b},$$

only one of which gives a real position.

(5) The condition $l = 0$, which makes one value of r^2 infinite and the other finite, corresponds to the case of a parabolic section, since in this case η and ζ in (3) Art. 232 are infinite, and therefore the centre of the section is at an infinite distance.

Cyclic Sections.

242. Although we have already determined the positions of the planes whose intersections with conicoids are circular, by treating such sections as particular cases of ellipses, it will be instructive to consider them from another point of view, since they have an interest peculiar to themselves in the solution of many problems both pure and physical.

243. *To find the cyclic sections of a conicoid central or non-central.*

Let the equation of the conicoid be

$$ax^2 + by^2 + cz^2 + 2dx + e = 0,$$

this may be written in the form

$$b(x^2 + y^2 + z^2) + (a-b)x^2 - (b-c)z^2 + 2dx + e = 0,$$

hence, if the conicoid be cut by a plane whose equation is

$$\sqrt{(a-b)}\,x \pm \sqrt{(b-c)}\,z = p\,\sqrt{(a-c)},$$

the coordinates of the points of the intersection satisfy the equation

$$b(x^2 + y^2 + z^2) + p\sqrt{(a-c)}\,\{\sqrt{(a-b)}\,x \mp \sqrt{(b-c)}\,z\} + 2dx + e = 0,$$

which is the equation of a sphere.

These plane sections will be real, if a, b, c are in order of magnitude when all are positive, if $a > b$ when c is negative, and if $c > b$, without regard to sign, when b and c are both negative. Also if $a = 0$, the sections are real when b and c are both positive, and $c > b$.

Hence, cyclic sections of central surfaces are parallel to the mean axis in the ellipsoid, to the greater transverse axis in the hyperboloid of one sheet, to the greater conjugate axis in the hyperboloid of two sheets.

If α be the inclination to the principal section (a, b),

$$\frac{\cos\alpha}{\sqrt{(a-b)}} = \frac{\sin\alpha}{\pm\sqrt{(b-c)}} = \frac{1}{\pm\sqrt{(a-c)}}.$$

Cyclic sections of the elliptic paraboloid are parallel to the tangent at the vertex of the principal section of greatest latus rectum; and for these sections putting $a = 0$,

$$\frac{\cos\alpha}{\sqrt{b}} = \frac{\sin\alpha}{\pm\sqrt{(c-b)}} = \frac{1}{\pm\sqrt{(c)}}.$$

It is obvious that these are the only cyclic sections, since a plane not parallel to one of the axes, as that of y, being of the form $lx + my + nz = p$, could not reduce the expresssion $(a-b)x^2 - (b-c)z^2$ to a linear form, so as to ensure that the points of intersection with the conicoid should lie on a sphere.

244. *Generation of a conicoid by the motion of a variable circle.*

From the last article it appears that the central conicoids and the elliptic paraboloid can be generated by the motion of a circle, the plane of which is parallel to either of two fixed planes, and the diameter of which changes so that it is always a chord of the section which is perpendicular to the line of intersection of the two fixed planes.

The centre of the circle of course describes in each case the diameter conjugate to the chords which it bisects.

245. Def. The point-circles in which the variable circle terminates are called *umbilics*, these are real only for the ellipsoid, the hyperboloid of two sheets and the elliptic paraboloid.

For the conicoid $ax^2 + by^2 + cz^2 = 1$, the four umbilics are given by $\pm \dfrac{ax}{\sqrt{(a-b)}} = \pm \dfrac{cz}{\sqrt{(b-c)}} = \sqrt{\left\{\dfrac{ac}{b(a-c)}\right\}}$, and $y = 0$, if $a > b > c$.

For the elliptic paraboloid, $by^2 + cz^2 = 2x$, $c > b$ the two umbilics at a finite distance are given by $\dfrac{cz}{\sqrt{(c-b)}} = \pm \dfrac{1}{\sqrt{(b)}}$, $2x = \dfrac{1}{b} - \dfrac{1}{c}$, and $y = 0$.

246. *Any two cyclic sections of opposite systems lie on one sphere.*

The equations of the planes of two cyclic sections of opposite systems are

$$\{\sqrt{(a-b)}\, x - \sqrt{(b-c)}\, z - k\} \{\sqrt{(a-b)}\, x + \sqrt{(b-c)}\, z - k'\} = 0;$$

or, $(a-b)\, x^2 - (b-c)\, z^2 - (k+k')\sqrt{(a-b)}\, x - (k-k')\sqrt{(b-c)}\, z + kk' = 0$.

Hence they intersect the surface in a sphere whose equation is

$$b\,(x^2 + y^2 + z^2) - 1 + (k+k')\sqrt{(a-b)}\, x + (k-k')\sqrt{(b-c)}\, z - kk' = 0.$$

247. It is an instructive problem to deduce the positions of the cyclic sections directly from the equation (4) obtained in Art. 230.

This equation may be written

$$a\lambda^2 + b\mu^2 + c\nu^2 = (\lambda^2 + \mu^2 + \nu^2)\,\rho,$$

and since, for a cyclic section, the values of r, and therefore of ρ, are equal for all values of λ, μ, ν consistent with the equation $l\lambda + m\mu + n\nu = 0$, it follows that

$$(\rho - a)(m\mu + n\nu)^2 + l^2 \{(\rho - b)\,\mu^2 + (\rho - c)\,\nu^2\} = 0$$

is true for an infinite number of values of $\mu : \nu$, the coefficients of μ^2, $\mu\nu$, and ν^2 are therefore each zero;

$$\therefore\ (\rho - a)\, mn = 0.$$

If $\rho = a$, either $l = 0$, or $\rho = b = c$, in which latter case the surface is spherical, and the equation is satisfied for any values of l, m, n, *i.e.* for any direction of the plane.

Also if $m = 0$, the coefficient of $\mu^2 = \rho - b = 0$, and similarly, for $n = 0$, $\rho = c$.

Hence, if the surface be not spherical, we must have l, m, or $n = 0$.

Suppose $m = 0$, then $u_0 = b$, and the coefficient of ν^2

$$= (\rho - a)\, n^2 + (\rho - c)\, l^2 = 0\,;$$

$$\therefore\ \frac{l^2}{b-a} = \frac{n^2}{c-b} = \frac{1}{c-a},$$

which give real values for l and n only under the same circumstances as are already stated in Art. 243.

The corresponding process for non-central surfaces can be followed out by the student.

248. *Geometrical investigation of the direction of a cyclic section of an ellipsoid.*

In the ellipsoid let OA, OB, OC be the semi-axes in order of magnitude, and if possible let a central circular section not pass through B, but cut AB and BC in P, Q respectively, $OP = OQ$, O being the centre of the section; but OP is intermediate between OA and OB in magnitude, and OQ between OB and OC, which is absurd; hence, the central circular section must contain the mean axis.

The inclination of the plane to OAB is the same as the angle ROA, OR being that radius vector of the section AC which $= OB$.

It is easy to give a similar proof for the hyperboloids.

249. Thus a method of obtaining the direction of the circular sections of an ellipsoid is to find the inclination to OA of a radius vector of the ellipse (a, c), whose length is b, the mean semi-axis of the ellipsoid; if α be this inclination,

$$\frac{b^2\cos^2\alpha}{a^2} + \frac{b^2\sin^2\alpha}{c^2} = 1 = \cos^2\alpha + \sin^2\alpha\,;$$

$$\therefore\ \frac{\sin^2\alpha}{\frac{1}{b^2}-\frac{1}{a^2}} = \frac{\cos^2\alpha}{\frac{1}{c^2}-\frac{1}{b^2}} = \frac{1}{\frac{1}{c^2}-\frac{1}{a^2}}.$$

XII.

(1) All spheres which intersect a given conicoid in plane sections and pass through a fixed point on it pass through one of two fixed circles.

(2) Find the equation of an ellipsoid referred to the planes of its central circular sections and a central plane at right angles to them. When these are rectangular axes, prove that the squares of the axes are in harmonical progression, and that the equation takes the form

$$\frac{(x+z)^2+y^2}{c^2}+\frac{(x-z)^2+y^2}{a^2}=2.$$

(3) Prove that through any point on an ellipsoid two planes of circular section can be drawn; but that when the circles are equal, the points must lie on one of the principal planes passing through the mean axis.

(4) If two circular sections of different systems be such that the sphere on which both lie is of constant radius mb, the locus of the centre of the sphere is the hyperbola $\dfrac{x^2}{a^2-b^2}-\dfrac{z^2}{b^2-c^2}=1-m^2$, $y=0$; a, b, c, being in descending order of magnitude.

(5) The sphere $(x^2+y^2+z^2+a^2-b^2-c^2)=2x\dfrac{\overline{a^2-b^2}\big]^{\frac{1}{2}}\,\overline{a^2-c^2}\big]^{\frac{1}{2}}}{a}$ meets the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ only at umbilics.

(6) The locus of centres of all plane sections of a given conicoid drawn through a given point is a similar and similarly situated conicoid, of which the given point and the centre of the given surface are extremities of a diameter.

(7) In a paraboloid of revolution, the eccentricity of any section is the cosine of the inclination of the plane to the axis of the surface, and the foci of the section are the points of contact with spheres inscribed in the surface.

(8) A sphere is described, having for a great circle a plane section of a given conicoid; prove that the plane of the circle in which it again meets the conicoid intersects the plane of the former circle in a straight line which lies in one of two fixed planes.

(9) A plane drawn through the origin perpendicular to any generating line of the cone $x^2(a^2-d^2)+y^2(b^2-d^2)+z^2(c^2-d^2)=0$, will intersect the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ in a section of constant area.

(10) In the hyperboloid $\dfrac{x^2}{a^2}+\dfrac{y^2-z^2}{c^2}=1$ $(a>c)$, the spheres, of which parallel circular sections are great circles, will have a common radical plane.

(11) On a central circular section of the ellipsoid $ax^2 + by^2 + cz^2 = 1$ a right circular cylinder is constructed, shew that if b be an arithmetic mean between a and c, the cylinder will again intersect the ellipsoid in an ellipse, the plane of which will be given by $(3a - c)\, x \pm (3c - a)\, z = 0$, and that the area of the ellipse will be $\frac{\pi}{b^2}\{2(a^2 + c^2) - 3b^2\}^{\frac{1}{2}}$.

(12) Prove that the difference of the squares of the axes of a central section is proportional to the product of the sines of the angles which it makes with the planes of circular section.

(13) Shew that if elliptic paraboloids have one of their cyclic sections coincident with a central cyclic section of $ax^2 + by^2 + cz^2 = 1$, a, b, c being in order of magnitude, the locus of their vertices will have the equations

$$\pm \frac{2x}{\sqrt{\{(b-a)(c-b)\}}} = \frac{z}{b-a} + \frac{1}{b(c-a)}\frac{1}{z}, \text{ and } y = 0.$$

Also, that the equation of the plane of the other cyclic section common to the conicoid and one of the paraboloids will be $\pm\sqrt{\left(\frac{c-b}{b-a}\right)}\, z + x + \frac{bl}{a} = 0$, where l is the latus rectum of any section parallel to the plane of xy.

(14) If sections of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ be made by planes passing through the centre, and through another given point $(x'y'z')$, the sections of greatest and least area will be at right angles to each other, and the areas will be $\frac{\pi abc}{r_1}$, $\frac{\pi abc}{r_2}$, r_1, r_2 being the semi-axes of the section made by the plane $\frac{xx'}{a} + \frac{yy'}{b} + \frac{zz'}{c} = 0$. Shew that the product of the areas will be constant if the point lie on the curve of intersection of the ellipsoid and a concentric sphere.

(15) The locus of the axes of sections of the surface $ax^2 + by^2 + cz^2 = 1$, which contain the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$, is the cone

$$(b - c)\, yz\, (mz - ny) + (c - a)\, zx\, (nx - lz) + (a - b)\, xy\, (ly - mx) = 0.$$

(16) Shew that the foci of all parabolic sections of the surface $\frac{y^2}{a} + \frac{z^2}{b} = x$, lie on the surface

$$\left(x - \frac{y^2}{a} - \frac{z^2}{b}\right)\left(\frac{y^2}{a} + \frac{z^2}{b}\right) = \frac{ab}{4}\left(\frac{y^2}{a^2} + \frac{z^2}{b^2}\right).$$

(17) Prove that the foci of all the centric sections of the conicoid $ax^2 + by^2 + cz^2 = 1$, lie on the surface

$$(x^2 + y^2 + z^2)(1 - ax^2 - by^2 - cz^2)\{a(c-b)^2 y^2z^2 + b(a-c)^2 z^2x^2 + c(b-a)^2 x^2y^2\}$$
$$= (ax^2 + by^2 + cz^2)\{(c-b)^2 y^2z^2 + (a-c)^2 z^2x^2 + (b-a)^2 x^2y^2\}.$$

CHAPTER XIII.

TANGENTS. CONICAL AND CYLINDRICAL ENVELOPES. NORMALS. CONJUGATE DIAMETERS.

250. On many accounts it is desirable that the student should be early acquainted with the chief properties connected with tangent lines and tangent planes to conicoids, before he is led to consider more general surfaces. We shall therefore give in this chapter some of the principal propositions relating to tangency in the case of the conicoids as represented by their equations in the simplest form. We shall also explain the properties of conjugate diameters and diametral planes.

Tangent Lines and Planes.

251. *To find the condition that a straight line shall touch a given conicoid, at a given point.*

Let the equation of the conicoid be

$$ax^2 + by^2 + cz^2 = 1,$$

the equations of a straight line drawn in a direction (λ, μ, ν) through the given point P, (f, g, h), are

$$\frac{x-f}{\lambda} = \frac{y-g}{\mu} = \frac{z-h}{\nu} = r, \qquad (1)$$

the values of r at the points P, Q, where it meets the conicoid, are given by the equation

$$a(f+\lambda r)^2 + b(g+\mu r)^2 + c(h+\nu r)^2 = 1;$$

or, since $af^2 + bg^2 + ch^2 = 1$,

$$2r(af\lambda + bg\mu + ch\nu) + r^2(a\lambda^2 + b\mu^2 + c\nu^2) = 0.$$

If the direction be such that Q coincides with P, the straight line will become a tangent, and in this case the two values of r will be zero; therefore

$$af\lambda + bg\mu + ch\nu = 0, \qquad (2)$$

is the condition of contact at the point (f, g, h).

252. *To find the equation of a tangent plane at a given point of a conicoid.*

The locus of all the tangent lines which can be drawn through the point (f, g, h), is found by eliminating λ, μ, ν between the equations (1) and (2) of the last article, giving

$$af(x-f) + bg(y-g) + ch(z-h) = 0,$$

$$\text{or } afx + bgy + chz = 1. \qquad (3)$$

The locus is therefore a plane, and this plane is called the *tangent plane* to the surface.

If p be the perpendicular from the centre upon the tangent plane

$$\frac{1}{p^2} = a^2f^2 + b^2g^2 + c^2h^2 \quad \text{(Art. 70)}.$$

Cor. 1. The generating lines of a hyperboloid of one sheet through the point (f, g, h) being two of the tangent lines, the tangent plane contains these lines, which together form what we have called the line-hyperbola in Art. 231.

Cor. 2. Since any generating line is intersected at every point by some line of the opposite system, no two of which lie in the same plane, it follows that the tangent plane to the hyperboloid at any point in a generating line changes its position as the point moves along the line.

253. *To find the equation of a tangent plane to a conicoid drawn in a given direction.*

Let (l, m, n) be the given direction of the normal to the tangent plane, so that its equation is $lx + my + nz = p$; comparing with the equation

$$afx + bgy + chz = 1,$$

$$\frac{l}{af} = \frac{m}{bg} = \frac{n}{ch} = p,$$

$$\text{and, since } af^2 + bg^2 + ch^2 = 1,$$

$$p^2 = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c},$$

and the equations of the two tangent planes in the given direction are determined.

254. The equation of a tangent plane to the cone $ax^2+by^2+cz^2=0$, is $a\lambda x+b\mu y+c\nu z=0$, if the line of contact be in direction (λ, μ, ν); and $lx+my+nz=0$ if the tangent plane be in direction (l, m, n), subject to the condition $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=0$.

255. *To find the equations of an asymptote to a central conicoid.*

Let the equation of the conicoid be $ax^2+by^2+cz^2=1$, and let (ξ, η, ζ) be any point in the asymptote whose equations are $\frac{x-\xi}{\lambda}=\frac{y-\eta}{\mu}=\frac{z-\zeta}{\nu}=r$, then the two values of r are infinite in the equation

$$a(\xi+\lambda r)^2+b(\eta+\mu r)^2+c(\zeta+\nu r)^2-1=0;$$

$$\therefore\ a\lambda^2+b\mu^2+c\nu^2=0, \text{ and } a\xi\lambda+b\eta\mu+c\zeta\nu=0,$$

and, if $a\xi^2+b\eta^2+c\zeta^2-1$ be not finite, the straight line lies entirely in the conicoid.

Hence, every straight line drawn in a tangent plane to the cone $ax^2+by^2+cz^2=0$, parallel to the line of contact, is an asymptote, including the generating lines in which it may intersect the conicoid.

256. *To find the nature of the intersection of a central conicoid with the tangent plane at a given point.*

Let the equation of the conicoid be $ax^2+by^2+cz^2=1$, that of the tangent plane at (f, g, h) is $afx+bgy+chz=1$, we have also $af^2+bg^2+ch^2=1$.

At the points of intersection

$$(af^2+bg^2)(ax^2+by^2)-(afx+bgy)^2=(1-ch^2)(1-cz^2)-(1-chz)^2;$$

$$\therefore\ ab(gx-fy)^2+c(z-h)^2=0.$$

For the ellipsoid and hyperboloid of two sheets the only solution is $\frac{x}{f}=\frac{y}{g}=\frac{z}{h}=1$, since ab, and c are of the same sign; for the hyperboloid of one sheet the section is two lines, since ab, and c are of contrary signs.

257. *To find the magnitude and direction of the axes of the section of a central conicoid made by a given plane through the centre.*

Let the equations of the conicoid and plane be

$$ax^2 + by^2 + cz^2 = 1, \text{ and } lx + my + nz = 0.$$

The equation of a sphere of radius r is $x^2 + y^2 + z^2 = r^2$; therefore the cone

$$(ar^2 - 1)\,x^2 + (br^2 - 1)\,y^2 + (cr^2 - 1)\,z^2 = 0$$

is the locus of all diameters of the conicoid which are of equal length $2r$; the cone, therefore, intersects the given plane in two lines which are the direction of equal diameters of the central section, and if r be chosen so that these directions coincide, the given plane will be a tangent plane to the cone, and the line of contact will be an axis of the section; therefore, by Art. 254, $\dfrac{l^2}{ar^2 - 1} + \dfrac{m^2}{br^2 - 1} + \dfrac{n^2}{cr^2 - 1} = 0$, which is the quadratic giving the lengths of the semi-axes. And, by the same article, if (λ, μ, ν) be the direction of the axis $2r$,

$$\frac{\lambda\,(ar^2 - 1)}{l} = \frac{\mu\,(br^2 - 1)}{m} = \frac{\nu\,(cr^2 - 1)}{n}.$$

258. *To find the locus of the points of contact of all tangent planes which pass through a given point external to a given conicoid.*

Let (f, g, h) be the given point, $ax^2 + by^2 + cz^2 = 1$, the equation of the conicoid.

The equation of a tangent plane at any point (ξ, η, ζ) on the conicoid is $a\xi x + b\eta y + c\zeta z = 1$, and if it pass through the given point $af\xi + bg\eta + ch\zeta = 1$.

The tangent planes at every point of the conicoid whose coordinates satisfy this equation pass through the given point, the locus required is therefore the section of the conicoid by the plane whose equation is $afx + bgy + chz = 1$.

259. Def. The plane containing the points of contact of all tangents from any point to a conicoid is the *Polar Plane*

of the point, and the point is the *Pole* of the plane, with respect to the conicoid.

This will be a definition whether the point be external or internal, if we consider that imaginary tangent planes have a real plane containing the imaginary curve of contact.

Another definition will be given which does not involve the consideration of tangency.

One of the most important propositions connecting the pole and polar plane is the following.

260. *If U be the polar plane of any point P with respect to a conicoid, the polar plane of any point Q in the plane U will pass through P.*

For, if $ax^2 + by^2 + cz^2 = 1$ be the equation of the conicoid, and (f, g, h) be the point P, the equation of its polar plane U will be

$$afx + bgy + chz = 1,$$

and, if (f', g', h') be any point Q in U,

$$aff' + bgg' + chh' = 1\,;$$

but the equation of the polar plane of Q is

$$af'x + bg'y + ch'z = 1,$$

which by the last equation contains (f, g, h), hence the polar plane of Q passes through P.

Conical and Cylindrical Envelopes.

261. *To find the conical envelope of a conicoid the vertex of the cone being a given point.*

If (f, g, h) be the given vertex, and (l, m, n) the direction of any generating line of the cone, the equation

$$a(f + lr)^2 + b(g + mr)^2 + c(h + nr)^2 = 1$$

must give equal values of r; therefore

$$(af^2 + bg^2 + ch^2 - 1)(al^2 + bm^2 + cn^2) = (afl + bgm + chn)^2.$$

If (x, y, z) be any point in the generating line,

$$\frac{x - f}{l} = \frac{y - g}{m} = \frac{z - h}{n},$$

hence the equation in l, m, n being homogeneous

$$(af^2+bg^2+ch^2-1)\{a(x-f)^2+b(y-g)^2+c(z-h)^2\}$$
$$=\{af(x-f)+bg(y-g)+ch(z-h)\}^2$$

is the equation of the conical envelope. This is readily reducible to the form

$$(af^2+bg^2+ch^2-1)(ax^2+by^2+cz^2-1)=(afx+bgy+chz-1)^2.$$

For, writing u, v, u_0 for $ax^2+by^2+cz^2-1$, $afx+bgy+chz-1$, and $af^2+bg^2+ch^2-1$ respectively, $u_0(u-2v+u_0)=(v-u_0)^2$; therefore $u_0u=v^2$.

262. This latter form may be obtained directly by Art. 258, since it is a surface of the second degree which passes through (f, g, h) and touches the conicoid where the plane $afx+bgy+chz=1$ cuts it.

For $Au=v^2$ is the equation of a surface which touches the conicoid $u=0$ where $v=0$, and, being satisfied by (f, g, h), we obtain, by substituting, $A=u_0$, since v becomes u_0.

263. *To find the equation of a cylinder, which envelopes a given central conicoid, and has its generating lines in a given direction.*

Let (λ, μ, ν) be the direction of the generating lines of the cylinder, and $ax^2+by^2+cz^2=1$ the equation of the conicoid.

The equations of a generating line through any point (ξ, η, ζ) of the cylinder are

$$\frac{x-\xi}{\lambda}=\frac{y-\eta}{\mu}=\frac{z-\zeta}{\nu}=r,$$

and where this line touches the conicoid the values of r, which are equal, are given by

$$a(\xi+\lambda r)^2+b(\eta+\mu r)^2+c(\zeta+\nu r)^2=1;$$

$$\therefore\ (a\lambda^2+b\mu^2+c\nu^2)(a\xi^2+b\eta^2+c\zeta^2-1)=(a\lambda\xi+b\mu\eta+c\nu\zeta)^2;$$

and, since (ξ, η, ζ) is any point on the cylinder, this is the equation of the enveloping cylinder.

This equation may also be deduced from that of the conical envelope by making (f, g, h) pass off to infinity in the direction (λ, μ, ν), so that $f:g:h=\lambda:\mu:\nu$.

Non-central Surfaces.

264. The corresponding propositions in the case of the non-central surfaces whose equations are of the form $by^2+cz^2=2x$ will be easily obtained by the student.

The condition for the tangency of $\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n}$ is

$$bgm+chn-l=0.$$

The equation of the tangent plane at (f, g, h) is

$$bgy+chz=x+f.$$

The equation of the tangent plane in direction (l, m, n) is

$$lx+my+nz=-\frac{1}{2l}\left(\frac{m^2}{b}+\frac{n^2}{c}\right).$$

The equation of the enveloping cone, vertex (f, g, h) is

$$(bg^2+ch^2-2f)\,\{b(y-g)^2+c(z-h)^2\}$$
$$=\{bg(y-g)+ch(z-h)-(x-f)\}^2,$$

or $$(bg^2+ch^2-2f)(by^2+cz^2-2x)=(bgy+chz-x-f)^2.$$

The equation of the enveloping cylinder, direction (λ, μ, ν), is

$$(b\mu^2+c\nu^2)(by^2+cz^2-2x)=(b\mu y+c\nu z-\lambda)^2.$$

Normals.

265. *To find the equations of the normal to a conicoid at any point.*

Def. A *normal* at a point is the straight line drawn perpendicular to the tangent plane at that point.

If (f, g, h) be the point, the equation of the tangent plane is $afx+bgy+chz=1$; therefore the equations of the normal will be

$$\frac{x-f}{af}=\frac{y-g}{bg}=\frac{z-h}{ch}=\pm rp,$$

if r be the distance between (x, y, z) and (f, g, h), and p be the perpendicular on the tangent plane from the centre.

266. *To shew that six normals can be drawn from a given point to a central conicoid.*

Let the equation of the conicoid be $ax^2+by^2+cz^2=1$.

The equations of a normal at a point (x, y, z) are

$$\frac{\xi - x}{ax} = \frac{\eta - y}{by} = \frac{\zeta - z}{cz} = \rho, \tag{1}$$

if this pass through a given point (f, g, h)

$$f = x(\rho a + 1),\quad g = y(\rho b + 1),\quad h = z(\rho c + 1);$$

$$\therefore\ \frac{af^2}{(\rho a + 1)^2} + \frac{bg^2}{(\rho b + 1)^2} + \frac{ch^2}{(\rho c + 1)^2} = 1,$$

which gives generally six values of ρ determining the feet of six normals from the given point.

267. *To shew that the locus of a point, from which three normals can be drawn to a central conicoid, which have their feet in a given plane section of the conicoid, is a straight line, and to find the condition to which the given plane section must be subject.**

Let the equation of the conicoid be $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and that of the given intersecting plane

$$l\frac{x}{a} + m\frac{y}{b} + n\frac{z}{c} = 1, \tag{1}$$

and let (ξ, η, ζ) be a point from which if six normals be drawn the feet of three of them will lie on the given plane section, the other three must then lie on some other plane section given by

$$l'\frac{x}{a} + m'\frac{y}{b} + n'\frac{z}{c} = d. \tag{2}$$

Hence the six feet lie on the surface

$$\left(l\frac{x}{a} + m\frac{y}{b} + n\frac{z}{c} - 1\right)\left(l'\frac{x}{a} + m'\frac{y}{b} + n'\frac{z}{c} - d\right)$$
$$- \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1. \tag{3}$$

And it is easily shewn by Art. 266 (1) that the six feet also lie on each of the three surfaces

$$U \equiv (b^2 - c^2)\,yz - b^2\eta z + c^2\zeta y = 0,$$
$$V \equiv (c^2 - a^2)\,zx - c^2\zeta x + a^2\xi z = 0,$$
$$W \equiv (a^2 - b^2)\,xy - a^2\xi y + b^2\eta x = 0,$$

* *Quarterly Journal*, vol. VIII., p. 69.

and therefore on the surface

$$\lambda U + \mu V + \nu W = 0. \tag{4}$$

Now we can make (3) and (4) identical by writing for the equation (2)

$$\frac{x}{la} + \frac{y}{mb} + \frac{z}{nc} + 1 = 0,$$

and equating the remainder of the coefficients, so that

$$\lambda (b^2 - c^2) = \left(\frac{m}{n} + \frac{n}{m}\right) \frac{1}{bc},$$

$$\mu (c^2 - a^2) = \left(\frac{n}{l} + \frac{l}{n}\right) \frac{1}{ca}, \tag{5}$$

$$\nu (a^2 - b^2) = \left(\frac{l}{m} + \frac{m}{l}\right) \frac{1}{ab},$$

$$\nu b^2 \eta - \mu c^2 \zeta = \left(l - \frac{1}{l}\right) \frac{1}{a},$$

$$\lambda c^2 \zeta - \nu a^2 \xi = \left(m - \frac{1}{m}\right) \frac{1}{b}, \tag{6}$$

$$\mu a^2 \xi - \lambda b^2 \eta = \left(n - \frac{1}{n}\right) \frac{1}{c}.$$

Hence it follows that, when the plane (1) is given, the locus of the point (ξ, η, ζ) is a straight line, since equations (6) are equivalent to two equations in ξ, η, ζ and a relation between l, m, n, which must be satisfied in order that normals at some three points of the plane section may meet in a point.

This equation of condition may be written

$$\frac{(m^2 + n^2)(l^2 - 1)}{b^2 - c^2} + \frac{(n^2 + l^2)(m^2 - 1)}{c^2 - a^2} + \frac{(l^2 + m^2)(n^2 - 1)}{a^2 - b^2} = 0;$$

or, as in Wolstenholme's *Problems*, 1150,

$$(m^2 n^2 - l^2)(b^2 - c^2)^2 + (n^2 l^2 - m^2)(c^2 - a)^2 + (l^2 m^2 - n^2)(a^2 - b^2)^2 = 0.$$

Since $\frac{l}{a}, \frac{m}{b}, \frac{n}{c}$, are the Boothian coordinates of the given plane, this gives the tangential equation of a surface fixed with reference to the conicoid, to which all the planes which satisfy the required condition must be tangents.

If the student desire to be acquainted with other elegant properties of the six normals, he should consult the paper by F. Purser referred to for this article.

Conjugate Diameters.

268. Def. A diametral plane of a conicoid is the locus of the middle points of a system of parallel chords.

269. *To find the diametral plane corresponding to a given series of parallel chords in a given central conicoid.*

Let the equation of the surface be

$$ax^2 + by^2 + cz^2 = 1,$$

λ, μ, ν the direction-cosines of each of a series of parallel chords, and (ξ, η, ζ) the middle point of any one of them.

The equation of this chord will be

$$\frac{x-\xi}{\lambda} = \frac{y-\eta}{\mu} = \frac{z-\zeta}{\nu} = r,$$

and we shall have, for the points in which it meets the surface, the equation

$$a(\xi + \lambda r)^2 + b(\eta + \mu r)^2 + c(\zeta + \nu r)^2 = 1.$$

But, since (ξ, η, ζ) is the middle point of the chord, the values of r obtained from this equation will be equal and of opposite signs, and therefore the equation

$$a\lambda\xi + b\mu\eta + c\nu\zeta = 0$$

will give the locus of the middle points of all such chords, which is the diametral plane required.

The form of this equation shews that it passes through the centre, as it manifestly ought to do.

We shall have, conversely, that any central plane whose equation is $lx + my + nz = 0$, will bisect a series of chords parallel to the straight line $\frac{ax}{l} = \frac{by}{m} = \frac{cz}{n}$, which is called the diameter *conjugate* to the plane. It appears from Art. 229 that the locus of the centres of a series of sections of the surface parallel to a given central plane is the diameter conjugate to that plane.

If a conicoid be referred to a diametral plane, as that of xy, and the corresponding conjugate diameter as the axis of z, since every straight line parallel to Oz will be bisected by the plane of xy, the equation of the surface can only contain even powers of z. Hence, if we can find three planes such that the intersection of any two is conjugate to the third, the equation of the surface referred to these planes will be of the form $ax^2+by^2+cz^2=1$. We proceed to investigate the conditions of the existence of such planes.

270. *To find the conditions that each of three central planes of a central conicoid may be diametral to the intersection of the other two.*

Let the direction-cosines of normals to the planes be $(l_1m_1n_1)$, $(l_2m_2n_2)$, and $(l_3m_3n_3)$.

The equations of the diameter conjugate to the first will be

$$\frac{ax}{l_1}=\frac{by}{m_1}=\frac{cz}{n_1},$$

and if this be in the intersection of the other two planes, and therefore lie in each of them, we shall have

$$l_2\frac{l_1}{a}+m_2\frac{m_1}{b}+n_2\frac{n_1}{c}=0, \text{ and } l_3\frac{l_1}{a}+m_3\frac{m_1}{b}+n_3\frac{n_1}{c}=0.$$

Hence, if the three conditions

$$\frac{l_2l_3}{a}+\frac{m_2m_3}{b}+\frac{n_2n_3}{c}=\frac{l_3l_1}{a}+\frac{m_3m_1}{b}+\frac{n_3n_1}{c}=\frac{l_1l_2}{a}+\frac{m_1m_2}{b}+\frac{n_1n_2}{c}=0$$

be satisfied, the planes will be such as required.

These planes are called *conjugate planes*, and their intersections *conjugate diameters*.

Since we have only three relations between the six quantities which determine the planes, there will be an infinite number of such systems, and we can determine such a system satisfying any three other relations which we may choose, provided the resulting equations are not inconsistent with those already obtained. For example, we can, in general, determine a system of conjugate planes each of which shall pass through one of three given points.

271. *To find the relations between the coordinates of the extremities of a system of conjugate diameters of a central conicoid.*

The equation of the surface being $ax^2+by^2+cz^2=1$, and the coordinates of the extremities of the semi-diameters r_1, r_2, r_3 being (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), we shall have, since the points lie on the surface,

$$ax_1^2+by_1^2+cz_1^2=ax_2^2+by_2^2+cz_2^2=ax_3^2+by_3^2+cz_3^2=1, \quad (1)$$

and, since the diameters through the points are conjugate,

$$\begin{aligned} ax_2x_3+by_2y_3+cz_2z_3 &= ax_3x_1+by_3y_1+cz_3z_1 \\ &= ax_1x_2+by_1y_2+cz_1z_2=0. \quad (2) \end{aligned}$$

The systems (1) and (2) shew that

$$x_1\sqrt{a},\ y_1\sqrt{b},\ z_1\sqrt{c};\quad x_2\sqrt{a},\ y_2\sqrt{b},\ z_2\sqrt{c};\quad x_3\sqrt{a},\ y_3\sqrt{b},\ z_3\sqrt{c};$$

are the direction-cosines of three straight lines at right angles to each other, and we know therefore (Art. 143) that they are equivalent to the systems

$$ax_1^2+ax_2^2+ax_3^2=by_1^2+by_2^2+by_3^2=cz_1^2+cz_2^2+cz_3^2=1,$$

$$y_1z_1+y_2z_2+y_3z_3=z_1x_1+z_2x_2+z_3x_3=x_1y_1+x_2y_2+x_3y_3=0. \quad (3)$$

Hence, in the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, we shall have

$$x_1^2+x_2^2+x_3^2=a^2,\quad y_1^2+y_2^2+y_3^2=b^2,\quad z_1^2+z_2^2+z_3^2=c^2, \quad (4)$$

or the sum of the squares of the projections of three conjugate diameters on one of the axes is equal to the square of that axis.

If (l, m, n) be the direction of any line, by (3) and (4)

$$\begin{aligned} (lx_1+my_1+nz_1)^2+(lx_2+my_2+nz_2)^2+(lx_3+my_3+nz_3)^2 \\ = l^2a^2+m^2b^2+n^2c^2=p^2, \end{aligned}$$

$$r_1^2-(lx_1+my_1+nz_1)^2+\ldots=a^2+b^2+c^2-p^2;$$

but $(lx_1+my_1+nz_1)^2$ and $r_1^2-(lx_1+my_1+nz_1)^2$ are respectively squares of the projections of r_1 upon a line and a plane whose directions are given by (l, m, n), hence it follows that

The sum of the squares of the projections of three conjugate diameters on any line or any plane is constant.

272. *To find the relations which exist between the lengths of a system of conjugate diameters of a central conicoid and the angles between them.*

Let the equation of the surface referred to its principal axes be $ax^2+by^2+cz^2=1$, and, referred to a system of conjugate diameters inclined at angles α, β, γ, let it be $a'x^2+b'y^2+c'z^2=1$. The invariants derived from $h(x^2+y^2+z^2)-ax^2-by^2-cz^2$, and the transformed expression, see Art. 153, give the equations

$$\frac{1}{a'}+\frac{1}{b'}+\frac{1}{c'}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}, \tag{1}$$

$$\frac{\sin^2\alpha}{b'c'}+\frac{\sin^2\beta}{c'a'}+\frac{\sin^2\gamma}{a'b'}=\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}, \tag{2}$$

and $$\frac{1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma}{a'b'c'}=\frac{1}{abc}. \tag{3}$$

When the surface is an ellipsoid, all these lengths are real, and we see from (1) that the sum of the squares of three conjugate radii is constant; from (2) that the sum of the squares of the faces of a parallelepiped having three conjugate radii as conterminous edges is constant; and from (3) that the volume of such a parallelepiped is constant.

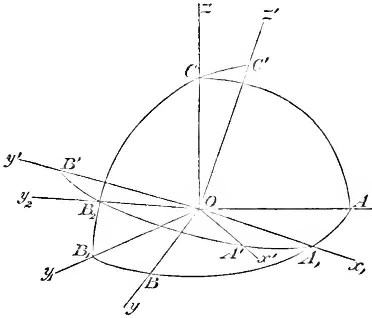
In the hyperboloid of one sheet, since abc is negative, and $1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma$ is always positive, $a'b'c'$ must be negative, but a', b', and c' cannot all be negative, hence one and only one is negative; that is, in a hyperboloid of one sheet, two of a system of conjugate diameters meet the surface in real points, and the third does not.

In the hyperboloid of two sheets, abc, and therefore $a'b'c'$ is positive, hence two, or none, of the three a', b', c' are negative. If none be negative, writing $\frac{1}{a^2}$, $-\frac{1}{b^2}$, $-\frac{1}{c^2}$ for a, b, c respectively, we must have both $a^2-b^2-c^2$ and $b^2c^2-a^2(b^2+c^2)$ positive, which are easily shewn to be inconsistent. Hence two must be negative; or, in the hyperboloid of two sheets, one and only one of a system of conjugate diameters meets the surface in real points.

273. The relations (1) and (3) may be obtained geometrically. We will give the proof in the case of the ellipsoid, and leave the other two cases as exercises for the student.

Let Ox, Oy, Oz be the directions of the axes of the surface,

Ox', Oy', Oz' those of a system of conjugate diameters; A, B, C, A', B', C', the extremities of the semi-diameters along those axes; a, b, c, a', b', c' their lengths. Also, let the sections AB, $A'B'$ intersect in A_1, let OB_2 be the semi-diameter conjugate to OA_1 in the section $A'B'$, and let CB_2 meet AB in B_1; $OA_1 = a_1$, $OB_1 = b_1$, $OB_2 = b_2$.



Then, the plane $A'B'$ being conjugate to OC', OA_1, OB_2, OC' will be a system of conjugate radii, or OA_1 will be conjugate to the plane $C'B_2$, and since OA_1 lies in a principal plane, $C'OB_2$ will be perpendicular to that plane, and will therefore contain OC; and OC, OB_1 being the principal semi-axes of the section B_1C', OA_1, OB_1, OC will be a system of conjugate diameters, and OA_1, OB_1 will be conjugate in the section AB.

Hence we have the equations

$$a_1^2 + b_2^2 = a'^2 + b'^2, \quad b_1^2 + c^2 = b_2^2 + c'^2, \quad a^2 + b^2 = a_1^2 + b_1^2,$$

and from these we obtain the relation

$$a'^2 + b'^2 + c'^2 = a^2 + b^2 + c^2.$$

Also, since the parallelogram of which OA', OB' are conterminous sides, is equal to that of which OA_1, OB_2 are conterminous sides, and similarly for the section B_1C', AB, we have

$$\text{vol}(a', b', c') = \text{vol}(a_1, b_2, c') = \text{vol}(a_1, b_1, c) = \text{vol}(a, b, c);$$

denoting any parallelepiped by three conterminous edges. This is equivalent to relation (3) of the last article.

274. *To find the diametral plane bisecting a given system of parallel chords, in the case of non-central conicoids.*

Let the equation of the surface be $\frac{y^2}{b} + \frac{z^2}{c} = 2x$, and let (λ, μ, ν) be the direction of the chords, the equation of the diametral plane will be

$$\mu\frac{y}{b} + \nu\frac{z}{c} = \lambda,$$

shewing that all the diametral planes are parallel to the common axis of the principal parabolic sections; a fact which might have been anticipated from the consideration that these surfaces have their centre on that axis at an infinite distance.

We cannot in these surfaces, as in the central conicoid, have a system of three conjugate planes at a finite distance, but we can find an infinite number, such that, for two of them, each bisects the chords parallel to the other and to a third plane. By taking the origin where the intersection of these two meets the paraboloid, and referring to these three planes, the equation of the surface will assume the form

$$\frac{y^2}{b'} + \frac{z^2}{c'} = 2x.$$

Let the equations of the two diametral planes be

$$m_1 y + n_1 z = 1, \qquad (1)$$

$$m_2 y + n_2 z = 1, \qquad (2)$$

and let the direction of the third plane be $(l_3 m_3 n_3)$. The direction-cosines of the chords bisected by (1) are in the ratios

$$1 : bm_1 : cn_1,$$

and if these be parallel to (2) and the third plane, we shall have

$$bm_1 m_2 + cn_1 n_2 = 0, \quad l_3 + bm_3 m_1 + cn_3 n_1 = 0.$$

Similarly, in order that (2) may be conjugate to the intersection of the other two, we shall have

$$bm_1 m_2 + cn_1 n_2 = 0, \quad l_3 + bm_2 m_3 + cn_2 n_3 = 0.$$

One of these is coincident with one of the former, and, there being thus only three relations necessary between the four quantities determining the planes, an infinite number of such systems can be determined.

Polar Plane.

275. We shall conclude this chapter by proving a theorem which gives rise to the definition of the polar plane of a point, alluded to in Art. 259.

DEF. The *polar plane* of any fixed point, with respect to a given conicoid, is a plane, which, with the conicoid, divides harmonically all straight lines passing through the fixed point.

276. *Through a fixed point a straight line is drawn meeting a central conicoid, and on this line a point is taken, such that its distance from the fixed point is a harmonic mean between the segments of the line made by the conicoid; to find the locus of the point.*

Let the equation of the conicoid be $ax^2 + by^2 + cz^2 = 1$, and let the equations of the straight line through the fixed point (f, g, h) be

$$\frac{x-f}{l} = \frac{y-g}{m} = \frac{z-h}{n} = r.$$

The values of r at the points of intersection are given by the equation

$$a(f+lr)^2 + b(g+mr)^2 + c(h+nr)^2 = 1.$$

If r_1, r_2 be the roots of this equation,

$$\frac{1}{r_1} + \frac{1}{r_2} = -\frac{2(afl + bgm + chn)}{af^2 + bg^2 + ch^2 - 1}.$$

If now r be taken for the distance from (f, g, h) of the point, whose locus is required, $\frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_2}$; therefore

$$af^2 + bg^2 + ch^2 - 1 + (afl + bgm + chn)\, r = 0,$$

and, since $lr = x - f$, &c., the equation of the locus will be

$$afx + bgy + chz = 1.$$

277. The corresponding locus for an arithmetic mean is a conicoid similar to the given one, of which a diameter is the line joining the given point and the centre.

For a geometric mean, the locus is a similar conicoid, which meets the given conicoid in the polar plane of the given point.

XIII.

(1) Find the equation of the tangent plane upon the principle that no other plane can pass between it and the surface in the neighbourhood of the point through which it is drawn.

(2) Prove that the three surfaces

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = \frac{2z}{c_1}, \quad \frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} = \frac{2z}{c_2}, \quad \frac{x^2}{a_3^2} + \frac{y^2}{b_3^2} = \frac{2z}{c_3},$$

will have a common tangent plane, if

$$\begin{vmatrix} a_1^2, & a_2^2, & a_3^2 \\ b_1^2, & b_2^2, & b_3^2 \\ c_1^2, & c_2^2, & c_3^2 \end{vmatrix} = 0.$$

(3) Tangent planes are drawn to an ellipsoid, and are such that their intersections with the plane zx are parallel to the line

$$cx\sqrt{(b^2 - c^2)} \pm az\sqrt{(a^2 - b^2)} = 0,$$

shew that the points of contact all lie on a circular section.

(4) The locus of the centres of sections of $ax^2 + by^2 + cz^2 = 1$ by planes which touch $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$, is

$$\frac{a^2}{\alpha}x^2 + \frac{b^2}{\beta}y^2 + \frac{c^2}{\gamma}z^2 = (ax^2 + by^2 + cz^2)^2.$$

(5) Prove that the tangent planes of the cone

$$\frac{x^2}{a^2(b^2 - c^2)} + \frac{y}{b^2(a^2 - c^2)} - \frac{z^2}{c^2(a^2 + b^2)} = 0,$$

cut the surface $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ in rectangular hyperbolas.

(6) Find the locus of the feet of the perpendiculars let fall from a point (α, β, γ) on the tangent planes to the cone $ax^2 + by^2 + cz^2 = 0$; and prove that if the locus be a plane curve it will be a circle, and that, if $b > a$ and c negative, the point must lie on one of the lines $y = 0$, $\dfrac{ax^2}{b - a} = \dfrac{cz^2}{c - b}$.

(7) The tangent planes to an ellipsoid at points lying on a plane section will intersect any fixed plane in straight lines which touch a conic section.

(8) Two similar and similarly situated ellipsoids have their axes in the ratio of $1 : n$, $n > 1$; from any point on the exterior as vertex a cone is drawn enveloping the interior, shew that the plane of the curve of intersection with the exterior ellipsoid touches another similar ellipsoid whose axes are to those of the interior as $n^2 - 2 : n$.

(9) If the area of the central curve in which a cylinder touches an ellipsoid be equal to that of the section containing the greatest and least axes, prove that the axis of the cylinder will lie on one of two planes.

(10) If an ellipsoid be placed on a horizontal plane with an axis $2c$ vertical, shew that the altitude of a star which will cast on the plane a circular shadow is $\tan^{-1}\dfrac{2c}{d}$, where d is the distance of the foci of the horizontal principal section.

(11) The normal at any point P of an ellipsoid meets the principal planes in G_1, G_2, G_3; prove that $PG_1 . PG_2 . PG_3$ varies as the cube of the area of the central section made by a plane conjugate to the diameter through P.

(12) If r be measured inwards along the normals to an ellipsoid so that $pr = m^2$ a constant, p being the perpendicular from the centre on the tangent plane, prove that the locus of the point thus obtained will be

$$\frac{a^2x^2}{(a^2 - m^2)^2} + \frac{b^2y^2}{(b^2 - m^2)^2} + \frac{c^2z^2}{(c^2 - m^2)^2} = 1.$$

What does the locus become when m is equal to one of the principal axes of the ellipsoid?

(13) The normals to the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ at points on the planes $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = \pm 1$ all intersect the straight line

$$ax (b^2 - c^2) = by (c^2 - a^2) = cz (a^2 - b^2).$$

(14) The enveloping cones which have as vertices two points on the same diameter of a conicoid intersect in two parallel planes between whose distances from the centre that of the tangent plane at the end of the diameter is a mean proportional.

(15) If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the extremities of three conjugate diameters of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, the equation of the plane passing through these points will be

$$\frac{x}{a^2} (x_1 + x_2 + x_3) + \frac{y}{b^2} (y_1 + y_2 + y_3) + \frac{z}{c^2} (z_1 + z_2 + z_3) = 1.$$

If one of the ends (x_1, y_1, z_1) be fixed, shew that the perpendicular from the centre on this plane will describe the cone

$$a^2x^2 + b^2y^2 + c^2z^2 = 3 (xx_1 + yy_1 + zz_1)^2.$$

(16) The locus of the centre of gravity of the triangle formed by joining the extremities of three conjugate diameters is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = \dfrac{1}{3}$, and the locus of the intersection of tangent planes drawn through their extremities is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 3$.

(17) Prove that the sum of the products of the perpendiculars from the two extremities of each of the three conjugate diameters of a conicoid upon any tangent plane is equal to twice the square of the perpendicular from the centre.

(18) Prove that the sum of the squares of the distances of any point of a given sphere from the six ends of any three conjugate diameters of a given concentric ellipsoid is invariable.

(19) If three straight lines be conjugate diameters, and the planes perpendicular to them conjugate planes, find their direction-cosines.

(20) The locus of points, from which rectilinear asymptotes can be drawn to the conicoid $ax^2 + by^2 + cz^2 = 1$, at right angles to each other, is the cone $a^2(b + c)x^2 + b^2(c + a)y^2 + c^2(a + b)z^2 = 0$.

(21) The locus of the intersection of two tangent planes to the cone $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$, which are at right angles, is the cone

$$(b + c)x^2 + (c + a)y^2 + (a + b)z^2 = 0.$$

(22) If two planes be drawn at right angles to each other touching the central conicoid $ax^2 + by^2 + cz^2 = 1$, and having their line of intersection in a given direction (l, m, n), shew that the locus of their line of intersection will be the right circular cylinder

$$x^2 + y^2 + z^2 = (lx + my + nz)^2 + \frac{m^2 + n^2}{a} + \frac{n^2 + l^2}{b} + \frac{l^2 + m^2}{c}.$$

(23) If the non-central conicoid $\frac{y^2}{a} + \frac{z^2}{b} = 2x$, be taken in the last problem, the locus will be $2l(lx + my + nz) - 2x = a(n^2 + l^2) + b(l^2 + m^2)$.

(24) The locus of the intersection of three tangent planes to the conicoid $ax^2 + by^2 + cz^2 = 1$, which are mutually at right angles, is

$$x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

and to the conicoid $\frac{y^2}{a} + \frac{z^2}{b} = 2x$, is $x = -\frac{a + b}{2}$.

(25) The locus of the intersection of three tangent lines to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, mutually at right angles, is

$$(b^2 + c^2)x^2 + (c^2 + a^2)y^2 + (a^2 + b^2)z^2 = b^2c^2 + c^2a^2 + a^2b^2.$$

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(26) Shew that the cone, whose vertex is (f, g, h), which envelopes the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, will be cut by the plane of xy in a rectangular

hyperbola, if the vertex lie on the spheroid $\frac{x^2 + y^2}{a^2 + b^2} + \frac{z^2}{c^2} = 1$. Also, if P be the centre of this section when V is the vertex, shew that the locus of P will be the inverse of the projection on the plane of xy of whatever curve V is made to describe upon the spheroid.

(27) A cone whose vertex is any point of the hyperbola $x = 0$, $\frac{z^2}{k^2} - \frac{y^2}{h^2} = 1$, envelopes the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, whose least semi-axis is c; and h and k satisfy the relation $b^2 - c^2 = \frac{b^2 (a^2 - b^2)}{h^2} + \frac{c^2 (a^2 - c^2)}{k^2}$; shew that the directrices of all the sections of the ellipsoid made by the planes of contact lie in one or other of two fixed planes.

(28) The points on a conicoid, the normals at which intersect the normal at a given point, all lie on a cone of the second degree having its vertex at the given point.

(29) The six normals drawn to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ from the point (f, g, h) all lie on the cone

$$(b^2 - c^2) \frac{f}{x - f} + (c^2 - a^2) \frac{g}{y - g} + (a^2 - b^2) \frac{h}{z - h} = 0.$$

(30) The six normals drawn to the conicoid $ax^2 + by^2 + cz^2 = 1$, from any point of the lines $a (b - c) x = \pm b (c - a) y = \pm c (a - b) z$, will lie on a cone of revolution.

(31) A section of the conicoid $ax^2 + by^2 + cz^2 = 1$ is made by a plane parallel to the axis of z, and the trace of the plane on xy is normal to the ellipse $\frac{ax^2}{(a - c)^2} + \frac{by^2}{(b - c)^2} = \frac{c^2}{(c^2 - ab)^2}$; prove that the normals to the conicoid at points in this plane will all intersect the same straight line.

(32) The normals to the paraboloid $\frac{y^2}{b} + \frac{z^2}{c} = 2x$, at points on the plane $px + qy + rz = 1$, will all meet one straight line if

$$p^3 (b - c) + 2p (q^2 b - r^2 c) = 2 \frac{(q^2 b + r^2 c)}{b - c}.$$

(33) Prove that a tangent plane to the cone $\frac{2x^2}{b - c} + \frac{y^2}{b} - \frac{z^2}{c} = 0$ will meet the paraboloid $\frac{y^2}{b} + \frac{z^2}{c} = 2x$ in points, the normals at which will all intersect in the same straight line, and the surface generated by the straight line will have for its equation

$$2 (b - c) \{x (by^2 - cz^2) + bc (y^2 - z^2)\}^2 = (by^2 - cz^2) (by^2 + cz^2)^2.$$

(34) Shew that any three equal conjugate diameters of an ellipsoid lie on the cone

$$\frac{x^2}{a^2}(2a^2 - b^2 - c^2) + \frac{y^2}{b^2}(2b^2 - c^2 - a^2) + \frac{z^2}{c^2}(2c^2 - a^2 - b^2) = 0,$$

and that the planes containing two of three equal conjugate diameters touch the cone

$$\frac{x^2}{a^2(2a^2 - b^2 - c^2)} + \frac{y^2}{b^2(2b^2 - c^2 - a^2)} + \frac{z^2}{c^2(2c^2 - a^2 - b^2)} = 0,$$

a, b, and c being the semi-axes of the ellipsoid.

(35) If α, β, γ be the angles between three equal conjugate radii of an ellipsoid, shew that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$, and $\cos\alpha \cos\beta \cos\gamma$, will be constant.

(36) The three acute angles made by any system of equal conjugate diameters of an ellipsoid will be together equal to two right angles, if $2(2a^2 - b^2 - c^2)(2b^2 - c^2 - a^2)(2c^2 - a^2 - b^2) = 27a^2b^2c^2$; $2a$, $2b$, $2c$, being the axes.

(37) In the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, shew that, if a^2 and b^2 be each $> c^2$, the equation of the surface may be put into the form $y^2 + z^2 - x^2 = d^2$, and if α, β, γ be the angles between (yz), (zx), (xy) in this case,

$$\cos\alpha(\cos\beta \cos\gamma - \cos\alpha) = \frac{(a^2 + b^2)(a^2 - c^2)(b^2 - c^2)}{2(a^2 + b^2 - c^2)^3}.$$

CHAPTER XIV.

CONFOCAL CONICOIDS. FOCAL CONICS. BIFOCAL CHORDS. CORRESPONDING POINTS.

278. In the preceding chapters we have considered the intersections of planes and straight lines with conicoids, in this chapter we shall discuss the mutual relations of conicoids grouped in a particular manner and called confocal conicoids, and prove certain theorems relating to their intersections, which will be useful hereafter when we treat of the curvature of surfaces and geodesic lines.

A knowledge of the whole theory of confocal surfaces is essential for the solution of many important problems in Physics; in fact, it was in the study of the attraction of ellipsoids that Maclaurin was first led to consider the properties of this class of surfaces.

The theory may be said to have been completed by Chasles,* although many valuable propositions are due to M'Cullagh.†

279. Def. Two conicoids are confocal, when the foci, real or imaginary, of their principal sections coincide; or, when the directions of their principal axes coincide, and their squares differ by a constant quantity.

Another definition will be afterwards given, but for our present purpose this definition has the advantage of greater simplicity.

If $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, and $\dfrac{x^2}{a'^2}+\dfrac{y^2}{b'^2}+\dfrac{z^2}{c'^2}=1$ be the equations of two confocal surfaces,

$$a^2-a'^2=b^2-b'^2=c^2-c'^2,$$

* Briot and Bouquet, *Géometrie Analytique*.

Aperçu Historique, L'Acad. Brux. 1837.

† With reference to the relative claims of Chasles and M'Cullagh to priority in certain investigations, see Liouville's *Journal*, vol. XI., p. 120, and *Proceedings of the Irish Academy*, vol. II., p. 501.

$$\text{or } a^2 - b^2 = a'^2 - b'^2 \text{ and } a^2 - c^2 = a'^2 - c'^2.$$

These relations have given rise to two methods of stating the equation of a group of confocal conicoids, called, for the sake of brevity, *confocals.*

In one method a is called the *primary* semi-axis, and the equation of the group is written

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - \beta^2} + \frac{z^2}{a^2 - \gamma^2} = 1,$$

β^2 and γ^2 being constant quantities, the individuals of the group being determined by assigning particular values to the primary axis.

In the other method the equation

$$\frac{x^2}{a^2 - k} + \frac{y^2}{b^2 - k} + \frac{z^2}{c^2 - k} = 1$$

represents all conicoids confocal with

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

by assigning arbitrary constant values to k.

280. *To shew that three conicoids can be drawn through a given point, confocal with a given central conicoid, and that these three will be an ellipsoid and the two hyperboloids.**

Let $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} + \dfrac{z^2}{\gamma^2} = 1$ be the given conicoid, in which $\alpha^2 > \beta^2 > \gamma^2$, where β^2 or γ^2 may be positive or negative, and let (f, g, h) be the given point.

All confocals are given by

$$\frac{x^2}{\alpha^2 - k} + \frac{y^2}{\beta^2 - k} + \frac{z^2}{\gamma^2 - k} = 1;$$

when any such passes through the point (f, g, h), k is determined by the equation

$$\frac{f^2}{\alpha^2 - k} + \frac{g^2}{\beta^2 - k} + \frac{h^2}{\gamma^2 - k} = 1,$$

* *Aper. Hist.* (30), p. 392; *Proc. Ir. Acad.*, vol. II., 196.

$$\text{or } (k-\alpha^2)(k-\beta^2)(k-\gamma^2)+f^2(k-\beta^2)(k-\gamma^2)$$
$$+g^2(k-\gamma^2)(k-\alpha^2)+h^2(k-\alpha^2)(k-\beta^2)=0.$$

If now we write for k in the left side of the equation, successively α^2, β^2, γ^2, $-\infty$, the signs of the result will be $+$, $-$, $+$, $-$; hence there are three real roots separated by these quantities.

Also, the quantities α^2-k, β^2-k, γ^2-k will have the following signs, corresponding to the three values of k,

$$\gamma^2>k, \quad + \quad + \quad +,$$
$$\beta^2>k>\gamma^2, \quad + \quad + \quad -,$$
$$\alpha^2>k>\beta^2, \quad + \quad - \quad -,$$

which proves the proposition.

COR. Two confocal ellipsoids or two confocal hyperboloids of the same kind cannot intersect.

281. *If two parallel tangent planes be drawn to two confocals, the difference of the squares of the perpendiculars from the common centre on these planes will be constant.**

For, if l, m, n be the direction cosines of the planes and p, p' be the two perpendiculars,

$$p^2=l^2a^2+m^2b^2+n^2c^2, \quad \text{(Art. 253)}$$
$$p'^2=l^2a'^2+m^2b'^2+n^2c'^2;$$
$$\therefore\ p^2\sim p'^2=l^2(a^2\sim a'^2)+m^2(b^2\sim b'^2)+n^2(c^2\sim c'^2)=a^2\sim a'^2.$$

COR. *If an ellipsoid and hyperboloid be confocal, all tangent planes to the ellipsoid drawn parallel to tangent planes to the conical asymptote of the hyperboloid will be at the same distance from the centre.*

For $p'=0$; $\therefore\ p^2=a^2-a'^2$.

282. *The poles of a given plane, taken with reference to each of a series of confocals, lie on a straight line perpendicular to this plane.†*

Let $\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2-\beta^2}+\dfrac{z^2}{a^2-\gamma^2}=1$ be the equation of any one of

* *Aper. Hist.* (37), p. 393; *Proc. Ir. Acad.*, vol. II., 491.

† *Aper. Hist.* (50), p. 397.

the surfaces, β and γ being constant, $\lambda x + \mu y + \nu z = 1$ that of the given plane; and let (ξ, η, ζ) be its pole with respect to the surface; therefore the equation of the plane must be the same as

$$\frac{\xi x}{a^2} + \frac{\eta y}{a^2 - \beta^2} + \frac{\zeta z}{a^2 - \gamma^2} = 1,$$

$$\text{or } \frac{\xi}{a^2} = \lambda, \quad \frac{\eta}{a^2 - \beta^2} = \mu, \quad \frac{\zeta}{a^2 - \gamma^2} = \nu;$$

$$\therefore \frac{\xi}{\lambda} - \frac{\eta}{\mu} = \beta^2, \quad \frac{\xi}{\lambda} - \frac{\zeta}{\nu} = \gamma^2;$$

these are the equations of the locus, which is evidently perpendicular to the given plane, and, since the point of contact of the particular confocal which touches the given plane is the pole with reference to that confocal, the locus is the normal at the point of contact to the confocal to which the given plane is a tangent.

Let (f, g, h) be the point of contact of the particular confocal which touches the given plane, 2α its primary axis, $2a$ that of any other confocal, and N the part of the normal intercepted between the plane and the polar plane of (f, g, h), whose equation is $\frac{fx}{a^2} + \frac{gy}{a^2 - \beta^2} + \frac{hz}{a^2 - \gamma^2} = 1$, which may be written

$$\frac{f(x-f)}{a^2} + \ldots = \frac{f^2}{\alpha^2} + \ldots - \frac{f^2}{a^2} + \ldots = (a^2 - \alpha^2)\left(\frac{f^2}{a^2\alpha^2} + \ldots\right),$$

and $x - f = -\frac{f}{\alpha^2} Np$, &c., N being measured inwards, and p being the perpendicular from the centre on the given plane;

$$\therefore Np = \alpha^2 - a^2.$$

Elliptic Coordinates.

283. *The position of a point on an ellipsoid is determined, when the octant on which it lies is known, by the primary axes of the confocal hyperboloids which pass through it.*

For if $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ be the equation of an ellipsoid and $a^2 - b^2 = \beta^2$, $a^2 - c^2 = \gamma^2$, $\gamma > \beta$, the primary axes of the confocal

hyperboloids, which pass through the point (ξ, η, ζ) on the ellipsoid, will be given by the equations

$$\frac{\xi^2}{\alpha^2} + \frac{\eta^2}{\alpha^2 - \beta^2} + \frac{\zeta^2}{\alpha^2 - \gamma^2} = 1,$$

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{a^2 - \beta^2} + \frac{\zeta^2}{a^2 - \gamma^2} = 1,$$

whence $$\frac{\xi^2}{a^2\alpha^2} + \frac{\eta^2}{(a^2 - \beta^2)(\alpha^2 - \beta^2)} + \frac{\zeta^2}{(a^2 - \gamma^2)(\alpha^2 - \gamma^2)} = 0,$$

$$\text{or } \alpha^4 + A\alpha^2 + \frac{\xi^2\beta^2\gamma^2}{a^2} = 0, \qquad (1)$$

if, then, a', a'' be the primary semi-axes of the two hyperboloids, the roots of the equation (1) are a'^2, a''^2;

$$\therefore\ a'^2a''^2 = \frac{\xi^2\beta^2\gamma^2}{a^2};$$

$$\therefore\ \xi^2 = \frac{a^2a'^2a''^2}{\beta^2\gamma^2} = \frac{a^2a'^2a''^2}{(a^2 - b^2)(a^2 - c^2)},$$

and similar expressions for η^2 and ζ^2.

DEF. The primary axes of the confocal hyperboloids, which pass through any point of an ellipsoid, are called the *elliptic coordinates* of that point.

It follows that the equations, in elliptic coordinates, of the two curves of intersection with the ellipsoid are a' or a'' = constant.

284. *When three confocals pass through a point, each of the normals to the confocals at this point is perpendicular to the other two.**

This will be proved, if we shew that at every point in the curve of intersection of two confocals the normals are at right angles.

Let the equations of two confocals be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

$$\text{and } \frac{x^2}{a^2 - k^2} + \frac{y^2}{b^2 - k^2} + \frac{z^2}{c^2 - k^2} = 1.$$

* *Aper. Hist.*, (30), p. 392.

If (ξ, η, ζ) be any point in the line of intersection, we find by subtracting

$$\frac{\xi^2}{a^2(a^2-k^2)} + \frac{\eta^2}{b^2(b^2-k^2)} + \frac{\zeta^2}{c^2(c^2-k^2)} = 0;$$

therefore if l, m, n and l', m', n' be the direction cosines of the normals

$$ll' + mm' + nn' = 0, \text{ (Art. 253)}$$

which proves the proposition.

285. *Three confocals pass through a point P, and a central section of one of them is made by a plane parallel to the tangent plane to it at P; to shew that the axes of this section are parallel to the normals at P to the other confocals. Also, if $2a$, $2a'$, $2a''$ be the lengths of the primary axes of the confocals, the squares of the semi-axes of the section will be $a^2 - a'^2$ and $a^2 - a''^2$.**

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $\frac{x^2}{a^2-k} + \frac{y^2}{b^2-k} + \frac{z^2}{c^2-k} = 1$ represent the three confocals through $P(f, g, h)$, by giving k the two values k', k'' derived from the quadratic

$$\frac{f^2}{a^2(a^2-k)} + \frac{g^2}{b^2(b^2-k)} + \frac{h^2}{c^2(c^2-k)} = 0. \qquad (1)$$

If (l, m, n), (l', m', n'), and (l'', m'', n'') be the directions of the normals at P to the three confocals

$$\frac{f}{a^2 l} = \frac{g}{b^2 m} = \frac{h}{c^2 n} \text{ and } \frac{f}{(a^2-k')\,l'} = \frac{g}{(b^2-k')\,m'} = \frac{h}{(c^2-k')\,n'}, \qquad (2)$$

and we shall obtain from (1) and (2) the equations

$$\frac{a^2l^2}{a^2-k'} + \frac{b^2m^2}{b^2-k'} + \frac{c^2n^2}{c^2-k'} = 0,$$

$$\text{and} \quad \frac{a^2-k'}{a^2}\cdot\frac{l'}{l} = \frac{b^2-k'}{b^2}\cdot\frac{m'}{m} = \frac{c^2-k'}{c^2}\cdot\frac{n'}{n}.$$

By Art. 234 or 257, k' is the square of one of the semi-axes of the section by the plane, $lx + my + nz = 0$, parallel to the tangent plane at P, and (l', m', n') is its direction; also, since $a^2 - k' = a'^2$, $a^2 - a'^2$ is the square of the semi-axis which is

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* *Proc. Ir. Acad.*, vol. II., p. 499.

parallel to the normal to the confocal corresponding to k', and similarly for k''.

If a', a'' belong to the hyperboloid of one and two sheets respectively, $a' > a''$, so that $a^2 - a''^2$ is the square of the smaller semi-axis.

COR. *When two confocals intersect, the normal to one of them at any point of the curve of intersection is parallel to an axis of a section made on the other by a plane parallel to the tangent plane at the point.*

*And diameters of each confocal, supposed central, drawn parallel to the normals to the other at every point of the curve of intersection are of constant length, being real for one and imaginary for the other.**

286. *To find the lengths of the perpendiculars from the centre upon the tangent planes at a given point to three confocals which pass through that point in terms of the primary axes.*

$$\text{Since } \frac{1}{p^2} = \frac{f^2}{a^4} + \frac{g^2}{b^4} + \frac{h^2}{c^4},$$

the equation $\dfrac{f^2}{a^2(a^2-k^2)} + \ldots = 0$ reduces to the form

$$k^4 - Ak^2 + \frac{a^2b^2c^2}{p^2} = 0;$$

$$\therefore\ p^2k'^2k''^2 = a^2b^2c^2 \text{ and } p^2 = \frac{a^2b^2c^2}{(a^2-a'^2)(a^2-a''^2)},\ p'^2 = \text{\&c.}$$

If, with the normals of the three confocals as axes, three new confocals be constructed, of which the semi-axes are respectively a, a', a''; b, b', b''; and c, c', c''; the squares of the coordinates of their points of intersection will be

$$\frac{a^2b^2c^2}{(a^2-a'^2)(a^2-a''^2)} = p^2, \text{ \&c. (Art. 283)};$$

they will therefore pass through the centre of the first three confocals, and the squares of the perpendiculars upon the tangent planes through that point will be $\dfrac{a^2a'^2a''^2}{(a^2-b^2)(a^2-c^2)} = \xi^2$, &c.;

* *Proc. Ir. Acad.*, vol. II., p. 499.

therefore the principal planes of the first three confocals will be tangent planes to the second three.*

287. *If p be the perpendicular on the tangent plane to an ellipsoid at any point of its curve of intersection with a confocal hyperboloid and d be the central radius parallel to the tangent to the curve, pd will be constant.*

$$\text{For } p^2 = \frac{a^2b^2c^2}{(a^2-a'^2)(a^2-a''^2)}.$$

Hence, if $2a'$ be the primary axis of the hyperboloid, since the tangent to the curve of intersection is a normal to the other confocal hyperboloid, $d^2 = a^2 - a''^2$ (Art. 285);

$$\therefore\ pd = \frac{abc}{(a^2-a'^2)^{\frac{1}{2}}} \text{ which is constant.}$$

288. *To find the directions of the principal axes of a cone which envelopes a given central conicoid.*

Let $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ be the given conicoid and (f, g, h) the vertex of the cone, then writing u_0 for $\dfrac{f^2}{a^2}+\dfrac{g^2}{b^2}+\dfrac{h^2}{c^2}-1$, the equation of the enveloping cone, referred to the vertex as origin, is

$$u_0\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) = \left(\frac{fx}{a^2}+\frac{gy}{b^2}+\frac{hz}{c^2}\right)^2.$$

The centre (ξ, η, ζ) of a section made by a plane $lx+my+nz=q$, v being written for $\dfrac{f\xi}{a^2}+\dfrac{g\eta}{b^2}+\dfrac{h\zeta}{c^2}$, is given, as in (3) Art. 229, by

$$\frac{u_0\xi - fv}{la^2} = \frac{u_0\eta - gv}{mb^2} = \frac{u_0\zeta - hv}{nc^2}$$

$$= \frac{u_0v-(u_0+1)v}{lf+mg+nh} = \frac{-v}{lf+mg+nh}.$$

Let $\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}+\dfrac{z^2}{\gamma^2}=1$ be the equation of a conicoid through

* *Aper. Hist.*, (36), p. 393.

(f, g, h), the tangent plane to it at that point being $lx+my+nz=p$ referred to the original axes;

$$\therefore\ p = lf + mg + nh \text{ and } fp = l\alpha^2 \ldots;$$

$$\therefore\ u_0\xi = \frac{lv}{p}(\alpha^2 - a^2), \text{ \&c.}$$

If $\alpha^2 - a^2 = \beta^2 - b^2 = \gamma^2 - c^2 = k$, or the conicoids be confocal,

$$\frac{\xi}{l} = \frac{\eta}{m} = \frac{\zeta}{n}.$$

Hence, the centres of all sections parallel to the tangent plane to any confocal through the vertex lie in the normal to that confocal, which is therefore a principal axis of the cone, or

The principal axes of a cone enveloping a given conicoid are normals to the three confocals drawn through the vertex.

289. *To find the equation of the enveloping cone referred to the normals to the confocals through the vertex as axes.*

Since these are principal axes the equation of the cone is of the form

$$Ax^2 + By^2 + Cz^2 = 0,$$

and, if p_1, p_2, p_3 be the perpendiculars from the centre of the conicoid on the tangent planes to the three confocals, the equation of the line joining the centre and vertex of the cone, referred to the same axes, will be $\dfrac{x}{p_1} = \dfrac{y}{p_2} = \dfrac{z}{p_3}$, and since this line passes through the centre of the curve of contact, the plane of contact will be parallel to the plane conjugate to this line, whose equation is

$$Ap_1x + Bp_2y + Cp_3z = 0 \text{ (Art. 269)},$$

but the equation of the plane of contact will be

$$\frac{x}{\lambda} + \frac{y}{\mu} + \frac{z}{\nu} = 1,$$

if λ, μ, ν be the intercepts on the three normals;

$$\therefore\ Ap_1\lambda = Bp_2\mu = Cp_3\nu,$$

hence, if a_1, a_2, a_3 be the primary semi-axes of the three con-

focals, since $p_1\lambda = a_1^2 - a^2$ &c. (Art. 282), the equation required will be

$$\frac{x^2}{a_1^2 - a^2} + \frac{y^2}{a_2^2 - a^2} + \frac{z^2}{a_3^2 - a^2} = 0.*$$

290. *To find the equation of the enveloped conicoid referred to the three normals through the vertex.*

Since the equation of the plane of contact is

$$\frac{p_1 x}{a_1^2 - a^2} + \frac{p_2 y}{a_2^2 - a^2} + \frac{p_3 z}{a_3^2 - a^2} = 1,$$

the equation of the conicoid is of the form

$$\rho\left(\frac{x^2}{a_1^2 - a^2} + \frac{y^2}{a_2^2 - a^2} + \frac{z^2}{a_3^2 - a^2}\right) + \left(\frac{p_1 x}{a_1^2 - a^2} + \frac{p_2 y}{a_2^2 - a^2} + \frac{p_3 z}{a_3^2 - a^2} - 1\right)^2 = 0;$$

if, therefore, we transform the origin to the centre $(-p_1, -p_2, -p_3)$, by writing $x - p_1$ for x, &c., the coefficients of x, y, z being equated to zero, we shall have

$$\rho + \frac{p_1^2}{a_1^2 - a^2} + \frac{p_2^2}{a_2^2 - a^2} + \frac{p_3^2}{a_3^2 - a^2} + 1 = 0,$$

hence the equation required will be

$$\left(\frac{p_1^2}{a_1^2 - a^2} + \frac{p_2^2}{a_2^2 - a^2} + \frac{p_3^2}{a_3^2 - a^2} + 1\right)\left(\frac{x^2}{a_1^2 - a^2} + \frac{y^2}{a_2^2 - a^2} + \frac{z^2}{a_3^2 - a^2}\right) = \left(\frac{p_1 x}{a_1^2 - a^2} + \frac{p_2 y}{a_2^2 - a^2} + \frac{p_3 z}{a_3^2 - a^2} - 1\right)^2.\dagger$$

291. *If from any point of a central conicoid a line be drawn touching two given confocals, the portion of this line intercepted between the point and the plane through the centre, parallel to the tangent plane at the point, will be constant.*‡

If, with P the given point as vertex, two cones be described enveloping the two confocals, the line under consideration will be one of their common sides.

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* Scott, *Quarterly Journal*, vol. VI., p. 258. † Ibid.

‡ *Proc. Ir. Acad.*, vol. II., p. 198.

Let $a_1 a_2 a_3$ be the primary semi-axes of the given conicoid and the two confocals drawn through the point P, α that of either of the conicoids to which the line through P is a tangent.

The equation of the cone enveloping the conicoid (α) referred to the normals to the confocals through P is

$$\frac{x^2}{a_1^2 - \alpha^2} + \frac{y^2}{a_2^2 - \alpha^2} + \frac{z^2}{a_3^2 - \alpha^2} = 0,$$

and $x = p_1$ is the equation of the central plane parallel to the tangent plane at P to (a_1), therefore the square of the portion of the common tangent intercepted between P and the central plane is $p_1^2 + y^2 + z^2$ where

$$\frac{p_1^2}{a^2 - \alpha^2} + \frac{y^2}{a_2^2 - \alpha^2} + \frac{z^2}{a_3^2 - \alpha^2} = 0,$$

in which the two values of α are a, a' the primary semi-axes of the two given confocals.

If $a_1^2 - \alpha^2 = k$, the equation may be written

$$\frac{p_1^2}{k} + \frac{y^2}{k - a_1^2 + a_2^2} + \frac{z^2}{k - a_1^2 + a_3^2} = 0,$$

$$\text{or } (p_1^2 + y^2 + z^2) k^2 + \ldots + p_1^2 (a_1^2 - a_2^2)(a_1^2 - a_3^2) = 0;$$

$$\therefore (a_1^2 - a^2)(a_1^2 - a'^2) = \frac{p_1^2 (a_1^2 - a_2^2)(a_1^2 - a_3^2)}{p_1^2 + y_2 + z^2},$$

$$\text{and } p_1^2 + y^2 + z^2 = \frac{p_1^2 (a_1^2 - a_2^2)(a_1^2 - a_3^2)}{(a_1^2 - a^2)(a_1^2 - a'^2)} = \frac{a_1^2 b_1^2 c_1^2}{(a_1^2 - a^2)(a_1^2 - a'^2)},$$

hence, the square of the intercepted portion is constant.

292. COR. *Two conicoids can be drawn confocal with a given conicoid and touching a given straight line.*

293. *If a chord of a given central conicoid touch two other surfaces confocal with it, the length of the chord will be proportional to the square of the diameter of the first surface parallel to it.**

Let a central section be taken containing the chord PP'; draw CQ a radius of this section parallel to $P'P$, and produce

* *Proc. Ir. Acad.*, vol. II., p. 198.

it to meet the tangent at P in T; let CN bisect PP', and PM, parallel to NC, meet CQ in M, then $CM.CT = CQ^2$; hence, since CT is constant by the last article, $PP' = 2CM \propto CQ^2$.

294. *When two confocals are viewed by an eye in any position, their apparent boundaries cut one another at right angles wherever they appear to intersect.**

The boundaries will appear to intersect in any line drawn from the eye so as to touch both surfaces.

Let the points of contact of such a line be P, (f, g, h) and P', (f', g', h'), and let the equations of the two confocals be

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - \beta^2} + \frac{z^2}{a^2 - \gamma^2} = 1 \text{ and } \frac{x^2}{a'^2} + \frac{y^2}{a'^2 - \beta^2} + \frac{z^2}{a'^2 - \gamma^2} = 1,$$

since PP' is a tangent line, the points P' and P are respectively in the tangent planes at P and P';

$$\therefore \frac{ff'}{a^2} + \frac{gg'}{a^2 - \beta^2} + \frac{hh'}{a^2 - \gamma^2} = 1,$$

$$\text{and } \frac{ff'}{a'^2} + \frac{gg'}{a'^2 - \beta^2} + \frac{hh'}{a'^2 - \gamma^2} = 1;$$

therefore, subtracting and dividing by $a^2 - a'^2$,

$$\frac{ff'}{a^2a'^2} + \frac{gg'}{(a^2 - \beta^2)(a'^2 - \beta^2)} + \frac{hh'}{(a^2 - \gamma^2)(a'^2 - \gamma^2)} = 0,$$

which shews that the tangent planes at P and P' are at right angles, and proves the proposition.

There may be four, two, or no apparent points of intersection, and, when they exist, they will be in the direction of the common generating lines of the two enveloping cones of which the eye is the common vertex.

295. The following method of dealing with tangents to confocal surfaces is due to Gilbert;† it enabled him to solve with great facility many of the problems in this subject.

* *Aper. Hist.*, (33), p. 392. *Proc. Ir. Acad.*, vol. II., p. 504.

† *Nouv. Annales*, vol. VI., p. 529.

He shews that, if we have points P, Q on two confocals whose equations are

$$\frac{x^2}{\lambda^2}+\frac{y^2}{\lambda^2-\beta^2}+\frac{z^2}{\lambda^2-\gamma^2}=1,$$

$$\text{and } \frac{x^2}{\theta^2}+\frac{y^2}{\theta^2-\beta^2}+\frac{z^2}{\theta^2-\gamma^2}=1,$$

and, if (θ, λ) denote the angle between the normals at P and Q, measured outwards, (δ, λ) the angle between $PQ\,(=\delta)$ and the normal at P, and $p_\lambda\, p_\theta$ be the perpendiculars from the centre on the tangent planes at P and Q respectively, then will

$$\cos(\theta, \lambda)=\frac{\delta}{\theta^2-\lambda^2}\{p_\theta \cos(\delta, \lambda)-p_\lambda \cos(\delta, \theta)\}.$$

Let x, y, z and x', y', z', be the coordinates of P and Q,

$$\cos(\delta, \lambda)=\frac{x'-x}{\delta}\cdot\frac{p_\lambda x}{\lambda^2}+\frac{y'-y}{\delta}\cdot\frac{p_\lambda y}{\lambda^2-\beta^2}+\frac{z'-z}{\delta}\cdot\frac{p_\lambda z}{\lambda^2-\gamma^2}\ \text{(Art. 253)}$$

$$\therefore\ \frac{\delta}{p_\lambda}\cos(\delta, \lambda)=\frac{xx'}{\lambda^2}+\frac{yy'}{\lambda^2-\beta^2}+\frac{zz'}{\lambda^2-\gamma^2}-1.$$

Similarly $$\frac{\delta}{p_\theta}\cos(\delta, \theta)=\frac{xx'}{\theta^2}+\frac{yy'}{\theta^2-\beta^2}+\frac{zz'}{\theta^2-\gamma^2}-1;$$

$$\therefore\ \frac{\delta}{p_\lambda}\cos(\delta, \lambda)-\frac{\delta}{p_\theta}\cos(\delta, \theta)$$

$$=(\theta^2-\lambda^2)\left\{\frac{xx'}{\theta^2\lambda^2}+\frac{yy'}{(\theta^2-\beta^2)(\lambda^2-\beta^2)}+\frac{zz'}{(\theta^2-\gamma^2)(\lambda^2-\gamma^2)}\right\},$$

$$\text{and } \cos(\theta, \lambda)=\frac{p_\lambda x}{\lambda^2}\cdot\frac{p_\theta x'}{\theta^2}+\ldots=\frac{\delta}{\theta^2-\lambda^2}\{p_\theta \cos(\delta, \lambda)-p_\lambda \cos(\delta, \theta)\}.$$

COR. If PQ be a tangent at Q to the surface (θ)

$$\cos(\delta, \theta)=0;\ \therefore\ \cos(\theta, \lambda)=\frac{\delta p_\theta}{\theta^2-\lambda^2}\cos(\delta, \lambda).$$

296. *If two confocals touch the same straight line, the tangent planes at the points of contact will be at right angles.*

For, if $\cos(\delta, \lambda)=0$ and $\cos(\delta, \theta)=0$, then $\cos(\theta, \lambda)=0$.

297. *If λ, μ, ν be the primary semi-axes of the confocals passing through P, θ that of a confocal enveloped by a cone of which*

P is the vertex, l, m, n the direction cosines of any generating line of the cone with reference to the normals to the three confocals,

$$\frac{l^2}{\theta^2-\lambda^2}+\frac{m^2}{\theta^2-\mu^2}+\frac{n^2}{\theta^2-\nu^2}=0.$$

For $\cos(\delta, \theta)=0$,

or $\cos(\delta, \lambda)\cos(\theta, \lambda)+\cos(\delta, \mu)\cos(\theta, \mu)+\cos(\delta, \nu)\cos(\theta, \nu)=0$;

$\therefore$ by Cor. Art. 295, $\dfrac{\cos^2(\delta, \lambda)}{\theta^2-\lambda^2}+\dfrac{\cos^2(\delta, \mu)}{\theta^2-\mu^2}+\dfrac{\cos^2(\delta, \nu)}{\theta^2-\nu^2}=0,$

which proves the proposition, and shews that the equation of the cone referred to the three normals is

$$\frac{x^2}{\theta^2-\lambda^2}+\frac{y^2}{\theta^2-\mu^2}+\frac{z^2}{\theta^2-\nu^2}=0.$$

298. *If any point P be taken in a fixed plane U, and on the normals to the three confocals passing through P lengths equal to the primary semi-axes be set off, the sum of the squares of the projections of these lengths on a normal to the plane U will be constant for all positions of P in that plane, viz. the square of the primary semi-axis of the confocal touching U.**

Let θ be the primary semi-axis of the confocal touching U, λ, μ, ν those of the three confocals, then

$$\cos(\delta, \lambda)\cos(\theta, \lambda)+\ldots=0;$$

$$\therefore (\theta^2-\lambda^2)\cos^2(\theta, \lambda)+(\theta^2-\mu^2)\cos^2(\theta, \mu)+(\theta^2-\nu^2)\cos^2(\theta, \nu)=0,$$

$$\text{or } \lambda^2\cos^2(\theta, \lambda)+\mu^2\cos^2(\theta, \mu)+\nu^2\cos^2(\theta, \nu)=\theta^2.$$

Focal Conics.

299. Among the surfaces of the system of confocals obtained by giving all values to k in the equation

$$\frac{x^2}{a^2-k}+\frac{y^2}{b^2-k}+\frac{z^2}{c^2-k}=1,$$

there are two which have a particular interest. If $a^2>b^2>c^2$ be all positive, suppose k to increase gradually from zero, the surfaces will change from ellipsoids to hyperboloids of one

* *Com. Rend.*, vol. XXII., p. 67.

sheet, as k passes through c^2, and from hyperboloids of one to those of two sheets as it passes through b^2.

When $k=c^2$, $z^2=0$, and the confocal may be considered as two planes coincident with that of xy, it being the limit of a very flat ellipsoid or hyperboloid of one sheet, as k is a little less or greater than c^2, the boundary of both being the ellipse

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1, \quad z=0.$$

In the same manner the hyperbola

$$\frac{x^2}{a^2-b^2}-\frac{z^2}{b^2-c^2}=1, \quad y=0,$$

is the boundary of the two flat hyperboloids of one and two sheets for which k is a little less and greater than b^2.

These conics are called the *focal ellipse* and *hyperbola* of any of the confocals; they pass through the foci of the two principal sections containing respectively the least and the mean axes of the ellipsoids of the system. The focal hyperbola also passes through the umbilici of the ellipsoids, for which

$$\frac{x^2}{(a^2-k)(a^2-b^2)}=\frac{z^2}{(c^2-k)(b^2-c^2)}=\frac{1}{a^2-c^2}.$$

But there are other properties which make the term focal conics peculiarly appropriate, and which we shall discuss in the next chapter.

300. *To find the confocal hyperboloids which pass through a point in the principal plane section of an ellipsoid which contains the primary and least axes.*

Let the equation of the ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-\beta^2}+\frac{z^2}{a^2-\gamma^2}=1,$$

and let $(f, 0, h)$ be the point in the plane of zx. If a' be the primary semi-axis of a confocal hyperboloid

$$\frac{f^2}{a^2a'^2}+\frac{0}{(a^2-\beta^2)(a'^2-\beta^2)}+\frac{h^2}{(a^2-\gamma^2)(a'^2-\gamma^2)}=0;$$

$$\text{therefore either } a'=\beta, \text{ or } a'^2=\frac{f^2\gamma^2}{a^2}.$$

The latter solution gives the hyperboloid

$$\frac{x^2}{\dfrac{f^2\gamma^2}{a^2}}+\frac{y^2}{\dfrac{f^2\gamma^2}{a^2}-\beta^2}-\frac{z^2}{\left(1-\dfrac{f^2}{a^2}\right)\gamma^2}=1,$$

which is a hyperboloid of one or two sheets according as $f >$ or $< \dfrac{a\beta}{\gamma}$, *i.e.* as the point is one side or the other of the focal hyperbola.

The other solution gives the focal hyperbola, which must be considered as a flat hyperboloid of two sheets or one, according to the position of the point.

301. We may observe here that as these *focal conics* belong to the group of confocals, many of the propositions given above can be applied to them. For example, a cone on a focal conic as base corresponds to an enveloping cone, since the focal conic is in this case the curve of contact of a flat ellipsoid enveloped by the cone; and the normals to the confocals through the vertex are axes of the cone.

302. *To find the locus of the vertices of all right cones which envelope a given conicoid.*

Since the positions of the principal axes of such cones, which are perpendicular to their axes of revolution, are indeterminate, we must consider three confocals through the vertex of some enveloping cone for which the directions of the normals to two of them will be indeterminate. It is evident that if we draw normals to a conicoid of which one of the axes is infinitely small, these normals will be parallel to that axis, unless the points at which they are drawn are indefinitely near the edge, and in passing round this edge from one side to the other the normals will assume every direction in a plane perpendicular to the tangent to the bounding focal conic, and this tangent being the normal to the third confocal will be the axis of a right cone. Hence, the vertices of right cones must lie in one of the focal conics.

303. COR. *The locus of the vertices of right cones on an elliptic base is an hyperbola in a plane perpendicular to its plane and vice versâ.*

For any ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ may be looked upon as the focal ellipse of a conicoid of which the focal hyperbola is $\frac{x^2}{\alpha^2}-\frac{z^2}{\gamma^2}=1$, if $a^2-\alpha^2=b^2=\gamma^2$, and the vertex must therefore lie in the focal hyperbola; hence the equation of the locus of the vertices is $\frac{x^2}{a^2-b^2}-\frac{z^2}{b^2}=1$.

Bifocal Chords.

304. Def. A *bifocal chord* of a conicoid is a chord which intersects two focal conics of the conicoid.

These conics being the limits of confocals of the conicoid, the properties proved in Arts. 291 and 293 are true for these two particular confocals, whence, if a bifocal chord be drawn through a point P on the conicoid, the portion intercepted between it and the diametral plane parallel to the tangent plane at P is the primary semi-axis of the conicoid, obtained by writing for a^2 and a'^2 in the formula $a_1^2-c_1^2$ and $a_1^2-b_1^2$; and the whole length of the chord is proportional to the square of the diameter parallel to it.

But in order to obtain a simple geometrical construction for the position of the four bifocal chords passing through a point, we have been obliged to deal with the problem in a direct manner, and we shall therefore give an independent solution of the problem concerning the intercepted lengths of the chords as a preliminary step.

305. *If P be one extremity of a bifocal chord of a conicoid, the portion of the chord intercepted between P and a plane through the centre, parallel to the tangent plane at P, will be equal to the primary semi-axes of the conicoid.*

Let l, m, n be the direction cosines of a bifocal chord drawn through a point P (f, g, h) of a conicoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, whose real focal conics are

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1 \quad\text{and}\quad \frac{x^2}{a^2-b^2}-\frac{z^2}{b^2-c^2}=1,$$

then $\dfrac{(nf-lh)^2}{a^2-c^2}+\dfrac{(ng-mh)^2}{b^2-c^2}=n^2$ (Art. 172, Cor. 1), (1)

and $\dfrac{(mf-lg)^2}{a^2-b^2}-\dfrac{(ng-mh)^2}{b^2-c^2}=m^2$; (2)

multiply (1) by $\dfrac{1}{c^2}-\dfrac{1}{a^2}$, and (2) by $\dfrac{1}{b^2}-\dfrac{1}{a^2}$, and add; then

$$\frac{(nf-lh)^2}{a^2c^2}+\frac{(mf-lg)^2}{a^2b^2}+\frac{(ng-mh)^2}{b^2c^2}=n^2\left(\frac{1}{c^2}-\frac{1}{a^2}\right)+m^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right),$$

or $$\left(\frac{f^2}{a^2}+\frac{g^2}{b^2}+\frac{h^2}{c^2}\right)\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)-\left(\frac{lf}{a^2}+\frac{mg}{b^2}+\frac{nh}{c^2}\right)^2$$
$$=\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}-\frac{1}{a^2};$$

$$\therefore\ \left(\frac{lf}{a^2}+\frac{mg}{b^2}+\frac{nh}{c^2}\right)^2=\frac{1}{a^2};$$

hence, if PG the normal at P, and PQ the bifocal chord, meet the central plane perpendicular to PG in F, E respectively,

$$a^2\cos^2 EPF=PF^2,\ \text{and}\ PF=PE\cos EPF;\ \therefore\ PE=a.$$

306. *If a tangent plane be drawn perpendicular to the bifocal chord, the distance from the centre to the point where the chord meets this plane will be equal to the primary semi-axis.*

This can be shewn by multiplying (1) by a^2-c^2, and (2) by a^2-b^2, and adding; whence

$$f^2+g^2+h^2-(lf+mg+nh)^2=a^2-l^2a^2-m^2b^2-n^2c^2.$$

307. *To shew that the four bifocal chords through any point P of an ellipsoid lie in two planes passing through the normal at P and intersecting the primary axis of the ellipsoid in the feet of the normals at the umbilics.*

The bifocal chords of an ellipsoid through any point $P(f, g, h)$ are the generating lines common to the conical surfaces, whose common vertex is at P, and whose guiding curves are the focal ellipse and hyperbola

$$\frac{x^2}{a^2-b^2}+\frac{y^2}{b^2-c^2}=1,\quad z=0,\quad\text{and}\quad \frac{x^2}{a^2-b^2}-\frac{z^2}{b^2-c^2}=1,\quad y=0.$$

Now the equations of the two cones referred to parallel axes through P, are

$$\frac{(fz-hx)^2}{a^2-c^2}+\frac{(gz-hy)^2}{b^2-c^2}=z^2, \text{ or } u=0, \tag{1}$$

$$\text{and } \frac{(fy-gx)^2}{a^2-b^2}-\frac{(gz-hy)^2}{b^2-c^2}=y^2, \text{ or } v=0; \tag{2}$$

and a common axis is the normal at P, whose equations are

$$\frac{a^2x}{f}=\frac{b^2y}{g}=\frac{c^2z}{h}.$$

The equation of two planes containing the bifocal lines and the normal is $\frac{u}{u_0}=\frac{v}{v_0}$, where u_0, v_0 are the values of u, v obtained by writing $\frac{f}{a^2}$, $\frac{g}{b^2}$, $\frac{h}{c^2}$ for x, y, z; and it is easily proved that $u_0 : v_0 :: \frac{h^2}{c^2} : \frac{g^2}{b^2}$.

Multiplying (1) by $\frac{g^2}{b^2}$, and (2) by $\frac{h^2}{c^2}$, and subtracting, we obtain

$$\frac{\left(\frac{x}{f}-\frac{y}{g}\right)^2}{c^2(a^2-b^2)}+\frac{\left(\frac{y}{g}-\frac{z}{h}\right)^2}{a^2(b^2-c^2)}+\frac{\left(\frac{z}{h}-\frac{x}{f}\right)^2}{b^2(c^2-a^2)}$$

$$=\frac{1}{(b^2-c^2)f^2}\left(\frac{c^2z^2}{b^2h^2}-\frac{2yz}{gh}+\frac{b^2y^2}{c^2g^2}\right),$$

$$\text{and } \left\{a^2(b^2-c^2)\frac{x}{f}+b^2(c^2-a^2)\frac{y}{g}+c^2(a^2-b^2)\frac{z}{h}\right\}^2$$

$$=(a^2-c^2)(a^2-b^2)\frac{a^2}{f^2}\left(\frac{c^2z}{h}-\frac{b^2y}{g}\right)^2.$$

Hence, the four bifocal lines lie in two planes whose equations referred to the axes of the ellipsoid are

$$a^2(b^2-c^2)\frac{x}{f}+b^2(c^2-a^2)\frac{y}{g}+c^2(a^2-b^2)\frac{z}{h}$$

$$=\pm\sqrt{(a^2-b^2)}\sqrt{(a^2-c^2)}\,\frac{a}{f}\left\{c^2\left(\frac{z}{h}-1\right)-b^2\left(\frac{y}{g}-1\right)\right\},$$

which pass through the points $\left\{\pm\frac{1}{a}\sqrt{(a^2-b^2)}\sqrt{(a^2-c^2)}, 0, 0\right\}$, that is, through the feet of the normals at the umbilics.

Corresponding Points.

308. DEF. If (x, y, z), (x', y', z') be two points P and P' situated respectively on the ellipsoids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \text{ and } \frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1,$$

P and P' are said to be corresponding points if $\frac{x}{a}=\frac{x'}{a'}$, $\frac{y}{b}=\frac{y'}{b'}$, $\frac{z}{c}=\frac{z'}{c'}$. Ivory first made use of points so connected in order to establish a relation between the attractions of an ellipsoid on an external and on an internal point, proving the following proposition:

309. *If P, Q be two points on an ellipsoid, and P', Q' the corresponding points on a confocal ellipsoid, $PQ' = P'Q$.*

Let (x, y, z), (ξ, η, ζ) be the points P, Q on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and let (x', y', z'), (ξ', η', ζ') be corresponding points P', Q' on the confocal $\frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1$.

Since $\frac{x}{a}=\frac{x'}{a'}$ and $\frac{\xi}{a}=\frac{\xi'}{a'}$, we have

$$(x-\xi')^2-(\xi-x')^2=\left(x-\frac{a'}{a}\xi\right)^2-\left(\xi-\frac{a'}{a}x\right)^2=(a^2-a'^2)\left(\frac{x^2}{a^2}-\frac{\xi^2}{a^2}\right),$$

and similarly for the other coordinates;

$$\therefore\ (x-\xi')^2+(y-\eta')^2+(z-\zeta')^2-\{(\xi-x')^2+(\eta-y')^2+(\zeta-z')^2\}=0,$$

$$\text{or } PQ'=P'Q.$$

310. Since $x^2-x'^2=(a^2-a'^2)\frac{x^2}{a^2}$, if O be the centre of the ellipsoid, then $OP^2-OP'^2=a^2-a'^2$.

311. *If any concentric ellipsoid be drawn through the vertex of a cone enveloping a given ellipsoid, the tangent plane at the point corresponding to the vertex will meet the second ellipsoid in an ellipse, every point of which will correspond to a point in the plane of contact; and, if the ellipsoids be confocal, the lengths of the tangents from the vertex will be equal to the corresponding radii of that ellipse.*

Let (f, g, h) be the vertex of the cone enveloping the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, (1), and let $\frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1$, (2), be an ellipsoid on which the vertex lies.

The plane of contact of the cone is $\frac{fx}{a^2}+\frac{gy}{b^2}+\frac{hz}{c^2}=1$ in which let (ξ, η, ζ) be any point, then, if (ξ', η', ζ') be the corresponding point on (2), and (f', g', h') correspond to the vertex, we have $f\xi = f'\xi'$;

$$\therefore \frac{f'\xi'}{a^2}+\frac{g'\eta'}{b^2}+\frac{h'\zeta'}{c^2}=1;$$

therefore (ξ', η', ζ') is on the plane which touches (1) at (f', g', h').

Also, if the ellipsoids be confocal, the latter part of the proposition is obvious by Ivory's theorem.

312. *If three points on an ellipsoid be the extremities of three conjugate diameters, the three corresponding points on any other ellipsoid will be also at the extremities of conjugate diameters.*

For the corresponding points on a concentric sphere are the same for both ellipsoids, and these are obviously at the extremities of three perpendicular radii.

313. *Confocal ellipsoids are cut by a fixed confocal hyperboloid; to shew that if any point be taken on the curve of intersection of one of the ellipsoids, the corresponding point on any other will lie on its curve of intersection.*

If (x, y, z) be a point on the intersection of $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ with the hyperboloid $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$ confocal with it,

$$(a^2-\alpha^2)\left(\frac{x^2}{a^2\alpha^2}+\frac{y^2}{b^2\beta^2}+\frac{z^2}{c^2\gamma^2}\right)=0;$$

if (x', y', z') be the point corresponding to (x, y, z) on the confocal ellipsoid $\frac{x'^2}{a'^2}+\frac{y'^2}{b'^2}+\frac{z'^2}{c'^2}=1$, writing $\frac{x'}{a'}$ for $\frac{x}{a}$, &c.,

$$\frac{x'^2}{a'^2\alpha^2}+\frac{y'^2}{b'^2\beta^2}+\frac{z'^2}{c'^2\gamma^2}=0;$$

therefore multiplying by $a'^2 - \alpha^2$, we obtain

$$\frac{x'^2}{\alpha^2} + \frac{y'^2}{\beta^2} + \frac{z'^2}{\gamma^2} = \frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} + \frac{z'^2}{c'^2} = 1\,;$$

therefore the point (x', y', z') also lies on the hyperboloid.

314. COR. If the hyperboloid which cuts the confocal ellipsoids be either of the flat hyperboloids, whose common edge is the focal hyperbola, all points on this edge will be corresponding points; so that the following theorem is proved:

The point on any ellipsoid which corresponds to an umbilic on a confocal ellipsoid will be itself an umbilic.

315. One of the series of ellipsoids in Art. 313 is the flat surface bounded by the focal ellipse, and the corresponding curve of intersection is the principal section of the hyperboloid, which is an ellipse or hyperbola as the hyperboloid is of one or two sheets, being confocal with the principal sections of the ellipsoids.

If hyperboloids of one and two sheets be confocal with the series of ellipsoids, α, α' their primary semi-axes will be elliptic coordinates of the points on any of the ellipsoids in which the curves of intersection with the hyperboloids intersect; and if r, r' be the distances of the corresponding point on the flat ellipsoid from the nearer and farther foci of the principal section of the ellipsoid, $r' + r = 2\alpha$ and $r' - r = 2\alpha'$, whence the plane curve corresponding to any curve on the ellipsoid, given in elliptic coordinates, can be found, or *vice versâ*. As an example, take the following:

316. *A curve is drawn on an ellipsoid such that, if a central section be taken parallel to the tangent plane at any point, the distance of the foci of the section will be constant, to find the corresponding curve on the plane of the focal ellipse.*

If α, α' be the elliptic coordinates of a point on the curve, the squares on the semi-axes of the section corresponding to this point will be $a^2 - \alpha^2$ and $a^2 - \alpha'^2$, hence $\alpha^2 - \alpha'^2$ is constant and the curve required will be $rr' =$ constant; if the given curve pass through the extremity of the least axis, the corresponding curve will be a lemniscate.

XIV.

(1) Prove that the locus of the points of intersection of tangent planes to three confocals, which are perpendicular to each other, is a sphere.

(2) If $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ be the equation of an ellipsoid, and (f, g, h) be any point, a', b', c', a'', b'', c'', a''', b''', c''' the semi-axes of the confocals through that point, then

$$\frac{f^2}{a^2a'^2a''^2a'''^2}+\frac{g^2}{b^2b'^2b''^2b'''^2}+\frac{h^2}{c^2c'^2c''^2c'''^2}=\frac{1}{a^2b^2c^2}.$$

(3) If normals be drawn from a fixed point to each of a series of confocals, shew that they will form a cone of the second degree.

(4) The points on a series of confocals, at which the normals are parallel, lie on an equilateral hyperbola of which one asymptote is parallel to the normals.

(5) If a, a', a'', a''', be the tranrsvese axes of an ellipsoid, and the three confocals which can be drawn through a given point, and if $a'^2+a''^2+a'''^2=3a^2$, then three tangent planes can be drawn from the given point to the given ellipsoid mutually at right angles.

(6) If through the vertex of a cone enveloping a given ellipsoid a confocal conicoid be drawn, the plane of contact will intersect the normal and the tangent plane to the conicoid at the vertex in a point and a line which are pole and polar with respect to the curve of contact; hence, shew that the normal is the axis of the cone.

(7) Through a straight line in one of the principal planes tangent planes are drawn to a series of confocal ellipsoids; prove that the points of contact lie on a plane, and that the normals at these points pass through a fixed point.

If a plane be drawn cutting the three principal planes, and through each of the lines of section tangent planes be drawn to the series of conicoids, prove that the three planes which are the loci of the points of contact will intersect in a straight line, which is perpendicular to the cutting plane and passes through the three fixed points in which the three series of normals intersect.

(8) The surface generated by the central circular sections of a system of confocal ellipsoids cuts the ellipsoids orthogonally.

(9) Through any fixed straight line tangent planes are drawn to each of a system of confocals, shew that the locus of the normals at the points of contact is a hyperbolic paraboloid.

(10) The rectangle contained by the side of a cone of revolution enveloping an ellipsoid, intercepted between the vertex and point of con-

tact, and the perpendicular from the centre upon the tangent plane at that point, is constant.

(11) PQ is a tangent at Q to a conicoid; λ, μ, ν are the primary semi-axes of the confocals through P; l, m, n the direction-cosines of PQ referred to the normals to the confocals at P; l', m', n' those of the perpendicular p from the centre on the tangent plane at Q; prove that, if $PQ = \delta$, $\lambda^2 ll' + \mu^2 mm' + \nu^2 nn' + p\delta = 0$.

(12) Shew that three non-central conicoids can be drawn through a given point, confocal with a given non-central, and that these will be a hyperbolic and two elliptic paraboloids. Shew also that the three normals to the confocals at the point are mutually at right angles.

(13) Three confocal paraboloids intersect in S; a cone having its vertex at S envelopes a fourth confocal paraboloid; find the equation of this cone referred to the normals to the confocals at S as axes.

(14) Prove that the polar of the foot of a normal to an ellipsoid with respect to the focal ellipse is the polar of the foot of the ordinate with respect to the principal section of the ellipsoid; also that the line joining the two feet is a normal to an ellipse similar to the principal section.

(15) When the two confocal hyperboloids through a point degenerate into flat surfaces bounded by the focal hyperbola, explain the perpendicularity of the three normals at the point.

(16) Find the three confocals of an ellipsoid through a point in one of the focal conics.

If $(f, 0, h)$ be a point on the focal hyperbola of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, shew that the ellipsoid which passes through it is

$$\frac{x^2}{(b^2 - c^2) f^2} + \frac{(a^2 - c^2) y^2}{(b^2 - c^2)^2 f^2 + (a^2 - b^2)^2 h^2} + \frac{z^2}{(a^2 - b^2) h^2} = \frac{a^2 - c^2}{(a^2 - b^2)(b^2 - c^2)}.$$

(17) a', b', c' and a'', b'', c'' are the semi-axes of the confocal hyperboloids which pass through a point P in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; p, p', p'' are the perpendiculars from the centre upon the tangent planes at P; shew that the equation of the plane of the focal ellipse referred to the three normals at P is $\frac{px}{c^2} + \frac{p'y}{c'^2} + \frac{p''z}{c''^2} + 1 = 0$, and that the bifocal lines lie in the two planes $\frac{a^2x^2}{p^2} = \frac{a'^2y^2}{p'^2}$.

(18) Shew that the equation of the circular cone containing the four bifocal lines through any point of an ellipsoid is $\left(\frac{a^2}{p^2} - 1\right) x^2 = y^2 + z^2$, referred to the three normals to the confocals through the point, p being the perpendicular on the tangent plane to the ellipsoid.

(19) If λ be the length of a bifocal chord of the paraboloid $\frac{y^2}{b} + \frac{z^2}{c} = x$, which makes angles β and γ with the axes of y and z respectively,

$$\frac{1}{\lambda} = \frac{\cos^2\beta}{b} + \frac{\cos^2\gamma}{c}.$$

(20) Find the locus of the point corresponding to a given point of an ellipsoid, on a system of confocal ellipsoids.

(21) Corresponding points on an ellipsoid of semi-axes a, b, c, and a sphere of radius r, being defined by the relations

$$\frac{x}{a} = \frac{x'}{r}, \quad \frac{y}{b} = \frac{y'}{r}, \quad \frac{z}{c} = \frac{z'}{r},$$

(x, y, z) being on the ellipsoid and (x', y', z') on the sphere, prove the following theorems:

(i) The points on a system of confocal ellipsoids corresponding to a fixed point on the sphere are on the intersection of two confocal hyperboloids.

(ii) The curve on the sphere corresponding to the curve of intersection of the ellipsoid and a confocal hyperboloid lies on the asymptotic cone of the hyperboloid.

(iii) If four curves on an ellipsoid of the kind described in ii, form a small rectangle of sides ds, ds', there will be a corresponding rectangle on the sphere whose sides $d\sigma$, $d\sigma'$ are connected with ds, ds' by the relations $rds = \lambda_2 d\sigma$, $rds' = \lambda_1 d\sigma'$, λ_1^2, λ_2^2 being the differences between the squares of the semi-axes of the ellipsoid and the two hyperboloids which intersect it in two adjacent sides of the small rectangle.

CHAPTER XV.

MODULAR AND UMBILICAL GENERATION OF CONICOIDS. PROPERTIES OF CONES AND SPHERO-CONICS.

317. THE modular and umbilical methods of generating conicoids, invented by MacCullagh and Salmon respectively, may be stated as follows:

For the *modular* method, "The locus of a point whose distance from a fixed point is in a constant ratio to its distance from a fixed straight line, *measured parallel to a fixed plane*, is a surface of the second degree."

The fixed point is called a *modular focus*, the fixed line a *directrix*, the constant ratio the *modulus*, and the plane the *directing plane*.

318. Since this locus contains *ten* disposable constants, viz. *three* dependent on the position of the fixed point, *four* on that of the fixed straight line, and *two* on the direction of the fixed plane, and *one* more, namely the constant ratio, the locus may, *in general*, be made to coincide with any surface which can be represented by an equation of the second degree in an infinite number of ways, since there will be only nine equations connecting the ten disposable constants.

If all but the three coordinates of the focus be eliminated, there will result two final equations determining a curve locus of such points; such curves are called *focal conics*, being the same as the limits of the confocals discussed in the last chapter.

Again, if all but the four constants which determine the position of the directrix be eliminated, there will be three final equations which, with the equations of the straight line, will determine a ruled surface, called a *dirigent cylinder*, the trace of which on the plane of the focal conic is called a *dirigent conic*.

319. For the *umbilical* method, "The locus of a point, the square of whose distance from a fixed point bears a constant ratio to the rectangle under its perpendicular distances from two directing planes, is a surface of the second degree."

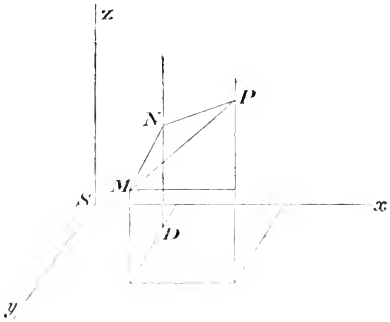
This locus contains ten disposable constants; *three* dependent on the position of the fixed point, *three* on the position of each of the directing planes, and *one* more, namely the constant ratio.

The fixed point, therefore, will not generally be unique, but may be on any point of a curve locus.

The fixed point is called an *umbilical focus*, the intersection of the planes a *directrix*, the constant ratio the *umbilical modulus*, and the locus of the focus the *umbilical focal conic*, the trace of the dirigent cylinder on the principal plane being the *umbilical dirigent conic.*

320. *To find the locus of a point whose distance from a focus is in a constant ratio to its distance from a directrix, measured parallel to a given directing plane.*

Let S the focus be taken for origin, Sz, Sy parallel to the directrix DN and the directing plane respectively, and let α, β be the coordinates of D in xy, e the modulus, and ω the angle of inclination of the directing plane to the plane of xy.



Let PN be drawn from the point (x, y, z) to the directrix parallel to the directing plane, NM parallel to Sy, and PM

perpendicular to NM; then PMN will be parallel to the directing plane. Hence we shall have $MN = y - \beta$, and $PM = (x - \alpha)\sec\omega$; and, P being a point in the locus, $SP = e.PN$;

$$\therefore\ x^2 + y^2 + z^2 = e^2\{(x-\alpha)^2\sec^2\omega + (y-\beta)^2\};$$

this is the equation of the locus required, which is a surface of the second order.

Since $z = x\tan\omega + h$ is the equation of any plane parallel to the directing plane, we have, at the points of intersection with the surface,

$$x^2\sec^2\omega + y^2 = x^2 + y^2 + (z-h)^2,$$

which, combined with the equation of the locus, shews that the curve of intersection lies on a sphere, except when $e = 1$, in which case it lies on another plane; hence, all sections parallel to the directing plane are circles, or when $e = 1$ straight lines.

321. That the section by a plane through S parallel to the directing plane is a circle is obvious geometrically, for, if this plane cut the directrix in H, the section is the locus of a point whose distances from S and H are in a constant ratio, and is therefore a circle, unless $e = 1$, in which case it is a straight line, and the surface is a hyperbolic paraboloid.

322. *To find the locus of a point, the square of whose distance from a focus is in a constant ratio to the rectangle under its distances from two fixed directing planes.*

Let the focus S be taken for the origin, the planes bisecting the angles between the directing planes being parallel to the planes of xy, yz.

Let also ω be the inclination of the directing planes to the plane of xy, α, γ the coordinates in the plane of zx of any point in the directrix, and e the constant ratio.

From any point P, let PQ, PR be drawn perpendicular to the directing planes;

$$\therefore\ SP^2 = ePQ.PR;$$

the equations of the directing planes will be

$$(x-\alpha)\sin\omega \pm (z-\gamma)\cos\omega = 0;$$

therefore, if x, y, z be the coordinates of P,

$$x^2 + y^2 + z^2 = e\{(x-\alpha)^2 \sin^2\omega - (z-\gamma)^2 \cos^2\omega\}$$

will be the equation of the locus, which is of the second degree.

If the surface be cut by a plane, parallel to either directing plane, whose equation is $(x-\alpha)\sin\omega \pm (z-\gamma)\cos\omega = p$, the curve of intersection will obviously lie on a sphere, and will therefore be a circle.

323. We have seen in both modes of generation, and it is also evident from the consideration of the number of constants, that the equation of a conicoid can be put into the form $S = UV$, where $U = 0$ and $V = 0$ are the equations of two real or imaginary planes, and $S = 0$ is the equation of a point sphere, or the imaginary cone having its vertex at a point which we have called a focus, and passing through the circle at infinity which is common to all spheres (Art. 227).

The conicoid and cone intersect in two plane curves crossing one another in two points P, Q which lie in the line of intersection of the planes U, V, called the directrix; a plane containing the tangent lines to the two curves at P will be a tangent plane to both conicoid and cone at P, and will therefore contain a tangent to the circle at infinity, which lies on the cone.

The generating line SP of the cone, whose vertex is the focus S, will be the intersection of two consecutive tangent planes to both conicoid and cone, each of which tangent planes contains a tangent line to the circle at infinity, and since the same argument holds for Q, SQ will be another such generating line.

If, therefore, a series of planes be drawn which touch both the conicoid and circle at infinity, these planes will envelope a torse, and a focus will be a point on the developable surface or torse in which two of its generating lines, which are not consecutive, intersect.

The locus of the foci will therefore lie on a double curve on the torse, and this curve will be the same for all conicoids enveloped by the same torse, touching also the circle at infinity.

Chasles suggested the following definition of confocals.

DEF. Conicoids are confocal when they are capable of being enveloped by the same developable surface described so as to touch the imaginary circle at infinity.

324. *To find the focal and dirigent conics in the case of central conicoids.*

Let (ξ, η, ζ) be a focus, and $(\xi', \eta', 0)$ the foot of the corresponding directrix supposed parallel to the axis of z.

The equation of the conicoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, whether generated by the modular or umbilical method, must coincide with the equation

$$(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 = \lambda\,(x-\xi')^2 + \mu\,(y-\eta')^2,$$

λ, μ being of the same or opposite signs. Comparing these equations, we have $\xi = \lambda\xi'$, $\eta = \mu\eta'$, $\zeta = 0$, and

$$(1-\lambda)\,a^2 = (1-\mu)\,b^2 = c^2 = \lambda\xi'^2 + \mu\eta'^2 - \xi^2 - \eta^2;$$

$$\therefore\ \lambda = 1 - \frac{c^2}{a^2},\quad \mu = 1 - \frac{c^2}{b^2},$$

$$\text{and } \lambda\xi'^2 - \xi^2 = \left(\frac{a^2}{a^2-c^2} - 1\right)\xi^2 = \frac{c^2}{a^2-c^2}\,\xi^2;$$

$$\therefore\ \frac{\xi^2}{a^2-c^2} + \frac{\eta^2}{b^2-c^2} = 1.$$

The focal conic is therefore confocal with the conicoid, and lies in the principal plane perpendicular to the directrix. Again, since

$$\xi = \frac{a^2-c^2}{a^2}\,\xi',\ \text{and}\ \eta = \frac{b^2-c^2}{b^2}\,\eta',$$

the equation of the dirigent cylinder is

$$(a^2-c^2)\,\frac{\xi'^2}{a^4} + (b^2-c^2)\,\frac{\eta'^2}{b^4} = 1.$$

325. *The focal and dirigent conics are reciprocals of each other with respect to the principal section in the plane of which they lie, and the line joining the foot of any directrix with the corresponding focus is a normal to the focal conic.*

The equation of a focal conic being $\frac{x^2}{a^2-c^2} + \frac{y^2}{b^2-c^2} = 1$, the

equation of the tangent at the point (ξ, η) is $\dfrac{x\xi}{a^2 - c^2} + \dfrac{y\eta}{b^2 - c^2} = 1$, or $\dfrac{x\xi'}{a^2} + \dfrac{y\eta'}{b^2} = 1$; whence it is the polar of (ξ', η'), the foot of the corresponding directrix, with respect to the section in xy.

Also, since $a^2(\xi' - \xi) = c^2\xi'$, and $b^2(\eta' - \eta) = c^2\eta'$, the equation of the tangent may be written

$$x(\xi' - \xi) + y(\eta' - \eta) = c^2;$$

it is therefore perpendicular to the line joining (ξ, η) and (ξ', η'), whence the second part of the proposition.

326. *If a section of a conicoid be made by a plane perpendicular to that of a focal conic, so that it contains a directrix, to shew that the distance of any point of the section from the directrix will have a constant ratio to the distance from the corresponding focus.*

For if $(\xi, \eta, 0)$ be the focus corresponding to the directrix (ξ', η'), the equation of the conicoid may be put into the form

$$(x - \xi)^2 + (y - \eta)^2 + z^2 = \lambda(x - \xi')^2 + \mu(y - \eta')^2,$$

and, if the equation of the plane be $\dfrac{x - \xi'}{l} = \dfrac{y - \eta'}{m} = r$, then for any point P of the section $(x - \xi)^2 + (y - \eta)^2 + z^2 = (\lambda l^2 + \mu m^2)\, r^2$; therefore $SP \propto PQ$, if PQ be perpendicular on the directrix.

327. COR. If the plane containing a directrix be perpendicular to the focal conic, the corresponding focus S will be a point in the plane (Art. 325), and will therefore be a focus of the section; hence, *Every point of a focal conic of a conicoid is a focus of the section made by a plane perpendicular to the focal conic at that point.*

328. *To find where a conicoid is intersected by its focal conics.*

If the directrix be parallel to Oz for the conicoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the equations of the focal conic will be $\dfrac{x^2}{a^2 - c^2} + \dfrac{y^2}{b^2 - c^2} = 1$, $z = 0$,

and it will intersect the conicoid if $\dfrac{x^2}{a^2(a^2-c^2)} + \dfrac{y^2}{b^2(b^2-c^2)} = 0$ give $x^2 : y^2$ positive.

Hence, if the focal conic and the corresponding principal section be both ellipses or both hyperbolas they do not intersect; but, if they be not of the same kind, they will intersect in the umbilics of the conicoid; whence the name umbilical focal conic.

Now, referring to the equations of Arts. 320 and 322, we see that in the modular method of generation the focus cannot lie on the conicoid, but may do so in the umbilical method, thus the umbilical focal conics correspond to the umbilical method of generation, and the other focal conics to the modular method.

329. *To find the focal and dirigent conics for non-central surfaces.*

For the paraboloids, comparing the equation

$$(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 = \lambda(x-\xi')^2 + \mu(y-\eta')^2$$

with $\dfrac{y^2}{b} + \dfrac{z^2}{c} = 2x$, we have $\lambda = 1$, $b(1-\mu) = c = \xi - \xi'$,

$$\eta = \mu\eta', \ \zeta = 0, \text{ and } \xi^2 + \eta^2 = \lambda\xi'^2 + \mu\eta'^2;$$

$$\therefore\ \mu = 1 - \frac{c}{b}, \text{ and } c(\xi + \xi') = \eta^2\left(\frac{b}{b-c} - 1\right);$$

$$\therefore\ \eta^2 = 2(b-c)(\xi - \tfrac{1}{2}c).$$

The focal conic is therefore a parabola, which has its vertex at the focus of the parabolic section parallel to the directrix, and is confocal with the section in its plane, since the abscissa of its focus is $\frac{1}{2}c + \frac{1}{2}(b-c) = \frac{1}{2}b$.

Also, the equation of the dirigent conic is $\eta'^2 = \dfrac{2b^2}{b-c}\left(\xi' + \dfrac{c}{2}\right)$, which is the reciprocal of the focal parabola with respect to the section $y^2 = 2bx$.

It will be found that the focal conic of an elliptic or hyperbolic cylinder is the two straight lines containing the foci of the principal sections; and that that of a parabolic cylinder is two straight lines, one of which is at an infinite distance and the other contains the foci of the principal sections.

330. In order to apply the modular method of generation to the investigation of properties of conicoids, the modulus and directing plane must be real, as well as the focal and dirigent conics, and, referring to Arts. 320 and 324, we obtain the following conditions:

I. For the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $a > b > c$,

$$(1 - e^2 \sec^2 \omega)\, a^2 = (1 - e^2)\, b^2 = c^2,$$

the only focal conic which is applicable being

$$\frac{x^2}{a^2 - c^2} + \frac{y^2}{b^2 - c^2} = 1.$$

For an oblate spheroid $a = b$ and $\omega = 0$. The prolate spheroid, for which $b = c$, cannot be generated by the modular method.

II. For the hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, $b > a$, the directing plane is parallel to Oy, and both the focal conics $\frac{x^2}{a^2 + c^2} + \frac{y^2}{b^2 + c^2} = 1$ and $\frac{y^2}{b^2 - a^2} - \frac{z^2}{c^2 + a^2} = 1$ are applicable, the corresponding moduli e, e' being given by $(e^2 - 1)\, b^2 = c^2$ and $(1 - e'^2)\, b^2 = a^2$, where

$$\frac{\cos^2 \omega}{e^2} + \frac{\sin^2 \omega}{e'^2} = 1,$$

so that the hyperboloid of one sheet can be generated by means of foci lying in a focal ellipse or a focal hyperbola, the greater modulus corresponding to the ellipse.

III. For the hyperboloid of two sheets $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, $b > c$, the directing plane is parallel to Oy, and the focal conic is $\frac{x^2}{a^2 + c^2} - \frac{y^2}{b^2 - c^2} = 1$, the modulus being given by $(1 - e^2)\, b^2 = c^2$.

The hyperboloid of revolution of two sheets, where $b = c$, cannot be constructed by the modular method.

IV. For the elliptic paraboloid $\frac{y^2}{b} + \frac{z^2}{c} = 2x$, $b > c$,

$$e = \cos \omega, \quad b\,(1 - e^2) = c = b \sin^2 \omega,$$

the focal conic is $y^2 = 2\,(b - c)\,(x - \tfrac{1}{2}c)$.

V. For the hyperbolic paraboloid $\dfrac{y^2}{b} - \dfrac{z^2}{c} = 2x$,

$$e = 1,\ b(1 - \sec^2\omega) = -c, \text{ or } b\tan^2\omega = c;$$

the focal parabolas are

$$y^2 = (b + c)(2x + c) \text{ and } z^2 = -(b + c)(2x - b),$$

each of which satisfies the modular method.

331. *To trace the changes of the surfaces and real focal conics corresponding to changes of the modulus from* 0 *to* ∞.

If we transfer the origin used in the equation of Art. 320 to the centre, and have regard to the sign of the constant term, we shall obtain the following results:

$e < \cos\omega$, Surface an ellipsoid, including an oblate spheroid.
Focal conic an ellipse.

$e = \cos\omega$, Surface an elliptic paraboloid.
Focal conic a parabola.

$e > \cos\omega$ and < 1, Surface at first an hyperboloid of two sheets, passing through a cone, to an hyperboloid of one sheet, conjugate axis perpendicular to the directrix.
Focal conic at first an hyperbola, transverse axis perpendicular to the directive axis, passing through the asymptotic limit, viz. two straight lines, to an hyperbola, transverse axis parallel to the directive axis.

$e = 1$, Surface an hyperbolic paraboloid.
Focal conics two parabolas.

$e > 1$, Surface an hyperboloid of one sheet, conjugate axis parallel to the directrix, including an hyperboloid of revolution.
Focal conic an ellipse, transverse axis parallel to the directive axis.

The hyperboloid of revolution of two sheets is lost between

$$e = 1 \text{ and } e = \cos\omega.$$

332. The directrix in the umbilical method of generation being parallel to the intersections of the two series of circular

sections, the plane of the focal conic is known, and by Art. 322 the umbilical modulus can be found in the same manner as in the modular method, and it will be seen that all surfaces can be generated, except the hyperboloid of one sheet, the hyperbolic paraboloid, and the oblate spheroid.

Properties of Conicoids deduced by the modular and umbilical methods.

333. *Every plane section of a conicoid, which is normal to a focal conic at any point, has that point for a focus.*

For, if S be the point through which the plane section passes, the corresponding directrix also will be in the plane; let PQ be perpendicular to this directrix from a point P on the section.

If the focal conic be modular, let PR parallel to a directing plane meet the directrix in R, then the ratios $SP : PR$ and $PR : PQ$ will be constant for every point in the section; and, therefore, $SP : PQ$ will be a constant ratio.

If the focal conic be umbilical, let PM, PN be perpendiculars on the planes through the directrix parallel to cyclic sections; then $SP^2 \propto PM.PN$, and for all points of the section $PM : PQ$ and $PN : PQ$ will be constant ratios, therefore $SP \propto PQ$.

334. *If a section of a conicoid be made by a plane perpendicular to the plane of a focal conic, it will contain two directrices; to shew that the sum or difference of the distances of any point of the section from the two corresponding foci will be constant.*

Let QD, $Q'D'$ be the two directrices, and S, S' the corresponding foci, which in this case will not be necessarily in the plane of the section; draw through any point P of the section QPQ' perpendicular to the directrices.

If the focal conic be modular, draw RPR' parallel to a directing plane meeting the directrices in R and R'.

Since the modulus is the same for both foci,

$$SP : PR :: S'P : PR';$$

$$\therefore\ SP : S'P :: PR : PR' :: PQ : PQ',$$

$$\text{and } SP \pm S'P : PQ \pm PQ' :: SP : PQ.$$

Now $PQ + PQ'$ or $PQ \sim PQ'$ is constant, according as P is or is not between the directrices, and $SP : PQ$ is constant, since $PR : PQ$ is so; therefore $SP \pm S'P$ is constant.

If the focal curve be umbilical, draw PM, PN perpendicular to the planes through QD parallel to the cyclic sections, and let PM', PN' be corresponding perpendiculars for $Q'D'$; then $PM : PQ :: PM' : PQ'$ and $PN : PQ :: PN' : PQ'$,

$$\text{also } SP^2 = e . PM . PN, \; S'P^2 = e . PM' . QN';$$

$$\therefore \; SP : S'P :: PQ : PQ',$$

and the argument proceeds as before.

335. *If a chord of a conicoid meet a directrix, the line joining the point in the directrix with the corresponding focus will bisect the angle between the focal distances of the extremities of the chord or its supplement.*

Let the chord PP' meet a directrix in Q, and let S be the corresponding focus, then $SP : PQ :: SP' : P'Q$;

$$\therefore \; SP : SP' :: PQ : P'Q,$$

which proves the proposition.

336. Cor. If PQ be a tangent to a conicoid at P, meeting a directrix in Q, and S be the corresponding focus, the angle PSQ will be a right angle.

337. *A straight line touching a conicoid makes equal angles with the lines drawn from the point of contact to the foci which correspond to the directrices which the line intersects.*

For, if P be the point of contact, Q, Q' the points in which the tangent meets the directrices, S, S' the foci, since the modulus is the same for both foci, we shall have $SP : PQ :: S'P : PQ'$.

Also the angles PSQ, $PS'Q'$ are right angles, therefore the triangles are similar, and the angles QPS, $Q'PS'$ are equal.

338. *If a cone, having its vertex in any directrix, envelope a conicoid, the plane of contact will pass through the corresponding focus, and be perpendicular to the line joining the focus with the vertex.*

If V be the vertex and S the focus, and VP be any side of the cone touching the surface in P, PSV will be a right angle. Hence the locus of P, which will be the curve of contact, will be in a plane through S perpendicular to VS.

339. *If the vertex of a cone be any point in a focal curve of a conicoid, and the base be any plane section of the conicoid, the line joining the vertex with the point in which the corresponding directrix meets the plane of section will be an axis of the cone.*

Let S be the vertex, and let the plane section cut the directrix in E, and EP, EP' be tangents to the section at P, P', then SP, SP' will be perpendicular to SE, the intersection of two tangent planes to the cone through SP, SP'; therefore SE will be an axis.

COR. 1. The second plane of section of the cone and conicoid will intersect the corresponding directrix in the same point as the first plane.

COR. 2. If the first plane of section pass through the directrix, the second will do so also, and in this case, since there will be an infinite number of axes of the cone, it will be one of revolution.

340. *If the vertex of an enveloping cone of a conicoid be a point on a focal conic of the conicoid, the cone will be one of revolution, and its internal axis will be the tangent to the focal curve at the vertex.*

Let V be the vertex of the cone, VP, VP' the tangents to the trace of the conicoid on the plane of the focal curve, then PP' will be a tangent to the dirigent conic at the foot of the corresponding directrix (Art. 325); and since the plane of contact is perpendicular to the plane of the focal curve, it will contain the corresponding directrix, the cone therefore will be one of revolution (Art. 339, Cor. 2).

Also, since the tangent at V to the focal conic is perpendicular to the directrix, and to the line joining V and the foot of the directrix (Art. 325), it will be perpendicular to the plane of circular section, and will be the internal axis of the cone.

Cones and Sphero-Conics.

The properties of cones of the second degree, and of their intersections with a sphere whose centre is at the vertex, called *sphero-conics*, have been discussed in an elaborate manner in two memoirs by Chasles.* In these investigations he has made use of certain reciprocal properties of the cyclic sections and focal lines, by which any theorem relating to cyclic sections involves a corresponding theorem concerning focal lines.

We can only make a selection of some of the innumerable propositions given by Chasles, in the proof of which we shall generally employ the properties of focal lines, in place of the reciprocal properties of the cyclic sections, employed with so much skill in those memoirs, for which we refer the student to a valuable translation by Graves.

341. *Focal conics of conical surfaces.*

Since a cone may be considered as the limit of either of the hyperboloids when the axes are made indefinitely small, if a, b, c be finite quantities proportional to the principal semi-axes of an hyperboloid, supposed indefinitely diminished, we obtain the equations of the cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$, $a>b$, and the corresponding focal conics, viz.

$$\frac{x^2}{a^2+c^2}+\frac{y^2}{b^2+c^2}=0,$$

$$\frac{y^2}{a^2-b^2}+\frac{z^2}{a^2+c^2}=0,$$

$$\text{and}\quad \frac{x^2}{a^2-b^2}-\frac{z^2}{b^2+c^2}=0.$$

The same consideration shews that *the line joining any focus with the foot of the corresponding directrix is perpendicular to the focal line containing the focus.*

The student should obtain these results by a direct comparison of the equation of the cone with such an equation as

$$(x-\xi)^2+y^2+(z-\zeta)^2=\lambda\,(x-\xi')^2+\nu\,(z-\zeta')^2,$$

* *Nouv. Mém. de l'Acad. Roy. de Bruxelles*, vol. VI.

which will give the focal and dirigent lines

$$\frac{\xi^2}{a^2 - b^2} - \frac{\zeta^2}{b^2 + c^2} = 0, \text{ and } \frac{a^2 - b^2}{a^4}\xi'^2 - \frac{b^2 + c^2}{c^4}\zeta'^2 = 0.$$

If he compare with the equation

$$(x - \xi)^2 + (y - \eta)^2 + z^2 = \lambda'(x - \xi')^2 + \mu'(y - \eta')^2,$$

he will obtain the equation

$$\frac{\xi^2}{a^2 + c^2} + \frac{\eta^2}{b^2 + c^2} = 0;$$

hence, when the directrix is in the axis, the vertex is a modular focus, λ and λ' are the squares of the moduli in the two cases, and are equal to $1 - \frac{b^2}{a^2}$ and $1 + \frac{c^2}{a^2}$.

The cone has therefore the property that all the three focal conics are real, having a common point in the vertex, two of them being ellipses evanescent in the transition between real and imaginary existence, and the third the limit of an hyperbola consisting of two right lines intersecting in the vertex.

The vertex is therefore not only modular, but *doubly* modular, since it is a point in two modular focal curves, and it is also an umbilical focus, as we see from the fact that the cone is the limit of two hyperboloids, for both of which the real focal hyperbola is modular, and for one the real focal ellipse is modular, while for the other it is umbilical.

Of the two moduli in the modular generation of the cone, the less modulus belongs to the focal lines, and is called by MacCullagh the *linear modulus*, while the other, to which only a single focus corresponds, is called the *singular modulus*.

342. *Cyclic sections of a cone.*

The equation of a cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$ may be written $x^2 + y^2 + z^2 = \left(1 + \frac{a^2}{c^2}\right)z^2 - \left(\frac{a^2}{b^2} - 1\right)y^2$; therefore, any plane section which is parallel to one of the planes $\left(1 + \frac{a^2}{c^2}\right)z^2 = \left(\frac{a^2}{b^2} - 1\right)y^2$, lies on a sphere and is circular.

The planes through the vertex, to which circular sections are parallel, are called *cyclic planes.*

343. *A sphere, which passes through the vertex of a cone and any circular section, touches the cyclic plane of the opposite system.*

A sphere can be described through any two circular sections parallel respectively to the two cyclic planes; let the plane of one of the circles approach indefinitely near to the vertex, in which case the circle degenerates into a point-circle lying on a cyclic plane, which is therefore a tangent plane to the sphere.

Conjugate Diameters of a Cone.

344. Take any line VA through the vertex of a cone, let VBC be its polar plane, and VAC any plane through VA intersecting VBC in VC, then VB the polar line of VAC will lie in VBC; also VC will be the polar line of the plane through VA, VB. Thus, if *any* plane cut VA, VB, VC in A, B, and C, the triangle ABC will be self-conjugate with respect to the section by the plane. If then a section be made by a plane parallel to VBC, the polar of the point in which this plane will cut VA will be at infinity, and the point will be the centre of the section. VA is therefore the locus of the centres of all sections by planes parallel to VBC; and VB, VC have the same relation to VAC, VAB respectively. VA, VB, VC, therefore, form a system of conjugate diameters of the cone.

Cor. If a plane cut a system of conjugate diametral planes of a cone, the triangle formed by the lines of intersection is self-conjugate with respect to the section of the cone by the plane.

Reciprocal Cones.

345. *If a cone be constructed whose sides are perpendicular to the tangent planes of any given cone, the tangent planes to it will be perpendicular to the sides of the given cone.*

Let any two tangent planes be drawn to a cone A, then two corresponding sides of the other cone B, perpendicular to those tangent planes, will be perpendicular to their line of intersection;

the line of intersection of the tangent planes to A is, therefore, perpendicular to the plane containing the corresponding sides of B.

Proceeding to the limit, the line of intersection becomes ultimately a side of the cone A, and the plane containing the sides of B a tangent plane to B; whence the truth of the proposition.

From this reciprocal property the cones are called *reciprocal cones.*

If $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ be the equation of a cone, $a^2x^2+b^2y^2-c^2z^2=0$ will be that of the reciprocal cone.

346. Any plane through the common vertex, having relations to one of the cones, has perpendicular to it a line which has reciprocal relations to the other cone, and the plane and line are said to *correspond.*

If two lines correspond respectively to two planes, they will each be perpendicular to the line of intersection of the planes, and the plane containing the two lines will correspond to the line of intersection of the two planes; also the angle between the planes will be equal to the angle between the corresponding lines.

347. The student will have no difficulty in establishing the following theorems:

To a line through the vertex of a cone and its polar plane with reference to the cone, correspond a plane and its polar line with reference to the reciprocal cone.

To three conjugate axes of a cone correspond three conjugate diametral planes of the reciprocal cone.

348. *The cyclic planes of a cone correspond to the focal lines of the reciprocal cone.*

The equation of the cyclic planes of the cone $a^2x^2+b^2y^2=c^2z^2$ is $(a^2-b^2)\,x^2=(b^2+c^2)\,z^2$, and that of the focal lines of the reciprocal cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$ is $\frac{x^2}{a^2-b^2}=\frac{z^2}{b^2+c^2}$; these focal lines are therefore perpendicular to the cyclic planes of the reciprocal cone.

The relation between the focal lines of one cone and the cyclic planes of the reciprocal cone is deduced geometrically thus:

To a cyclic plane corresponds a line VS perpendicular to it; any two conjugate axes in the cyclic plane are at right angles; therefore any two conjugate diametral planes of the reciprocal cone through VS are at right angles.

Let a plane be drawn perpendicular to VS through any point S, this plane will meet the two diametral planes in two perpendicular lines, and, by Art. 344, Cor., the pole of one of these lines with respect to the section of the cone will lie on the other line; therefore this pole is on the directrix, and S is the focus of the conic section; VS is therefore a focal line, having the focal property proved for any conicoid in Art. 333.

The locus of these directrices is called a *dirigent plane* by McCullagh and a *director plane* by Chasles, and this plane, with the perpendicular planes through the focal line, form a system of conjugate planes; it corresponds, therefore, with the third axis, conjugate to two perpendicular lines in a cyclic section, which contains the centres of circular sections parallel to that cyclic section.

349. The method of dealing with propositions connected with focal lines, by the modular and umbilical methods, may be seen by the following, in which we shall state the reciprocal theorems.

Properties of Cones of the Second Degree.

350. *The sines of the angles, which any side of a cone makes with a focal line and the corresponding dirigent plane, are in a constant ratio.*

Let a plane pass through any directrix DQ and the corresponding focus S, and let P be any point in the section of the cone made by this plane; V the vertex of the cone.

Draw PR, PQ perpendicular to the dirigent plane and directrix.

Then $SP : PQ$ and $PR : PQ$, and therefore $SP : PR$ are constant ratios; and DS being perpendicular to VS, VS is perpendicular to the plane of section, and PSV is a right angle.

Hence, the ratio of the sines proposed is $\frac{PS}{PV} : \frac{PR}{PV}$, and is therefore constant.

Reciprocal theorem. The ratio of the sines of the angles made by tangent planes with a cyclic plane and with the polar line of this cyclic plane is constant.

351. *The product of the sines of the angles which any side of a cone makes with the directing or cyclic planes is constant.*

If V be the vertex of a cone, P any point on the cone, PL, PL' perpendicular on the directing planes through V, then by the umbilical generation of the cone (Art. 322) PV^2 is proportional to $PL.PL'$; or $\frac{PL}{PV} \cdot \frac{PL'}{PV}$ is constant, which is the property enunciated.

Reciprocal Theorem. The product of the sines of the angles which each tangent plane to a cone makes with the two focal lines is constant.

352. *A tangent plane to a cone makes equal angles with the planes through the side of contact and each of the focal lines.*

For, let the tangent QPQ' perpendicular to the side VP meet the dirigent planes in the points Q, Q', and take S, S' the foci corresponding to the directrices through Q, Q'; then SQ is perpendicular to VS, and also to PS, and therefore to the plane VPS; also VP is perpendicular to SQ and PQ, and therefore to SP; hence SPQ is the inclination of the planes VPS, VPQ, and being equal to $S'PQ'$ (Art. 337), the proposition is proved.

Reciprocal Theorem. A tangent plane to a cone intersects the two cyclic planes in two straight lines, which make equal angles with the side of the cone along which it is touched by the tangent plane.

353. The last theorems are particular cases of the two following:

The planes passing through the two focal lines of a cone, and through the intersection of two tangent planes to the cone, make equal angles with these tangent planes.

And the reciprocal theorem:

A plane containing two sides of a cone intersects the cyclic planes in two straight lines, which respectively make equal angles with the two sides.

We give Chasles' proof of the reciprocal theorem as a good example of the geometrical treatment of problems connected with cyclic planes.

Take two circular sections of opposite systems of the cone, the plane of the two sides cuts the planes of the two circles in two chords, which, with the portions of the sides of the cone intercepted, form a quadrilateral inscribed in the circle in which the sphere containing the circular sections is cut by the plane of the two sides; two opposite angles of this quadrilateral are supplementary, hence the chords make equal angles with the sides of the cone, and, since they are parallel to the sections by the cyclic planes, the theorem is proved.

354. Simple propositions for the circle can be transformed into others relating to the cone with the same facility as in plane geometry properties of conics are obtained.

This is effected by considering the lines and points in the circle as the intersections of planes and straight lines, passing through the vertex of a cone, with the plane which cuts the cone in this circle.

It will be sufficient to give two examples of this transformation.

355. Two tangents to a circle make equal angles with the chord which joins the two points of contact, hence

Two tangent planes to a cone and the plane of the two sides of contact intersect a cyclic plane in three straight lines, the third of which bisects the angle between the other two.

The reciprocal theorem is,

If planes be drawn through a focal line of a cone, and two sides of the cone, and through the line of intersection of two

planes touching the cone along these sides, the third plane will bisect the angle between the first two.

356. Two tangents to a circle make equal angles with the line joining their point of intersection with the centre of the circle, hence

Two tangent planes to a cone, and the plane passing through their line of intersection, and through the conjugate of a cyclic plane, meet that cyclic plane in three lines, one of which bisects the angle between the other two.

The reciprocal theorem is,

The planes passing through a focal line of a cone and two sides of the cone make equal angles with the plane passing through the same focal line and the straight line in which the plane containing the two sides intersects the dirigent plane.

Sphero-conics.

357. If a cone of the second degree be cut by a sphere whose centre is at the vertex of the cone, the complete curve of intersection will be two closed curves, which will be plane curves if the cone be one of revolution.

Chasles observes that we obtain three distinct curves if we consider the portions of the complete curve of intersection contained on the three hemispheres cut off by the three principal planes of the cone.

First, consider the hemisphere whose base is perpendicular to the interior or principal axis of the cone, the figure is then a closed curve, and may be called a *spherical ellipse*, the foci of which are the points where the focal lines cut the hemisphere, having, it will be seen, properties in all respects corresponding to the foci of a plane ellipse.

Secondly, consider the hemisphere whose base is the other principal plane perpendicular to that containing the focal lines, the figure is then composed of two halves of spherical ellipses, which may together be called a *spherical hyperbola* whose foci lie within the concave portions, and it will be seen that sections of the sphere by the cyclic planes have properties similar to those of asymptotes.

Thirdly, consider the hemisphere whose base is the plane

containing the focal lines, the figure is then formed by two halves of spherical ellipses and has four foci and a centre where the minor axis of the cone meets the hemisphere.

We shall consider a sphero-conic to be one of the first two of these curves, viz. the spherical ellipse or hyperbola.

The curves in which a sphere cuts two reciprocal cones, of which its centre is the common vertex are called *reciprocal sphero-conics.*

The principal reciprocal property connecting the two may be stated thus:

Every point of a sphero-conic is the pole of a great circle which touches the reciprocal sphero-conic.

358. *The intersection of a central conicoid with a concentric sphere is a sphero-conic.*

For, if $ax^2 + by^2 + cz^2 = 1$ and $x^2 + y^2 + z^2 = r^2$ be their equations, the curve of intersection lies on the cone

$$(ar^2 - 1)\,x^2 + (br^2 - 1)\,y^2 + (cr^2 - 1)\,z^2 = 0\,;$$

this cone is evidently concyclic with the conicoid.

359. We give below two or three of the numerous properties of sphero-conics, which are the counterparts of properties of plane conics, each of which has its duplicate obtained by forming the reciprocal proposition. The proofs of these can be gathered from the previous articles; but as exercises in spherical trigonometry the student may take almost any ordinary property in plane conics relating to foci and directrices, and to asymptotes of hyperbolas, which correspond to the cyclic arcs, and find analogues to them in sphero-conics; he may also find equations corresponding to the polar equation of a conic or of a tangent to a conic, or of the auxiliary circle, or of the locus of the intersection of perpendicular tangents.

A tangent to a sphero-conic makes equal angles with the radii vectores drawn from the foci to the point of contact (Art 352).

An arc of a great circle which touches a sphero-conic and is cut off by the cyclic arcs is bisected at the point of contact.

The sum or difference of two radii drawn from the foci to any point of a sphero-conic is constant.

The sum or difference of the angles which a tangent to a sphero-conic makes with the cyclic arcs is constant.

The first theorem is proved by limits as in plane conics.

The product of the sines of arcs drawn perpendicular to the cyclic arcs from any point of a sphero-conic is constant (Art. 351).

The product of the sines of arcs drawn from the foci at right angles to a tangent to a sphero-conic is constant.

We give the following as an example of the mode of applying spherical trigonometry.

360. *The locus of the intersection of perpendicular tangents to a sphero-conic is another sphero-conic for which the product of the cosines of the distances from the foci of the first sphero-conic is constant.*

Let tangents at P, P' intersect at right angles in Q, 2α the major axis, 2γ the distance of the foci S, S', $SP=r$, $SP'=r'$, $SQ=\rho$, $S'Q=\rho'$, $\angle SQP=\angle S'QP'=\psi$, then

$$\cos 2\gamma=\cos\rho\,\cos\rho'+\sin\rho\,\sin\rho'\,\sin 2\psi,$$

and if p, p' be perpendiculars on PQ from S, S'

$$\sin p=\sin\psi\,\sin\rho,\quad \sin p'=\cos\psi\,\sin\rho',$$

and $\sin p\,\sin p'$, being constant, is equal to $\sin(\alpha-\gamma)\,\sin(\alpha+\gamma)$;

$$\therefore\ \cos\rho\,\cos\rho'=\cos 2\gamma-2\,\sin(\alpha-\gamma)\,\sin(\alpha+\gamma)=\cos 2\alpha,$$

from which the equation of the cone determining the sphero-conic may be easily deduced, viz.

$$(c^2-b^2)\,x^2+(c^2-a^2)\,y^2-(a^2+b^2)\,z^2=0.$$

This equation may also be obtained as follows:

The equation of two tangent planes to a cone $ax^2+by^2+cz^2=0$, drawn through a point (ξ,η,ζ) is

$$(a\xi^2+b\eta^2+c\zeta^2)\,(ax^2+by^2+cz^2)-(a\xi x+b\eta y+c\zeta z)^2=0,$$

$$\text{or}\ \ (lx+my+nz)\,(l'x+m'y+n'z)=0,$$

$$\therefore\ \frac{ll'}{a\,(b\eta^2+c\zeta^2)}=\frac{mm'}{b\,(c\zeta^2+a\xi^2)}=\frac{nn'}{c\,(a\xi^2+b\eta^2)},$$

and, when the tangent planes are at right angles,

$$a\,(b\eta^2+c\zeta^2)+b\,(c\zeta^2+a\xi^2)+c\,(a\xi^2+b\eta^2)=0\,;$$

$$\therefore\ \left(\frac{1}{b}+\frac{1}{c}\right)\xi^2+\left(\frac{1}{c}+\frac{1}{a}\right)\eta^2+\left(\frac{1}{a}+\frac{1}{b}\right)\zeta^2=0,$$

$$\text{or}\ \frac{\xi^2}{a}+\frac{\eta^2}{b}+\frac{\zeta^2}{c}=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(\xi^2+\eta^2+\zeta^2).$$

XV.

(1) Prove that the equation of two circular sections of an ellipsoid in elliptic coordinates is $a'^2 + a''^2 - \frac{2d}{d_0} a'a'' = (a^2 - c^2) \frac{r^2}{b^2}$, where r is the radius of each circle, d the distance of its centre from that of the ellipsoid, and d_0 the central distance of the umbilic.

(2) Prove that the points on the plane of the focal ellipse of an ellipsoid which correspond to those of a circular section lie also in a circle, whose area is to that of the section as $a^2 - c^2 : b^2$.

(3) The plane of any cyclic section of an ellipsoid will intersect the dirigent cylinder in an ellipse similar to the principal section in which the focal conic lies; if the plane touch at an umbilic, the umbilic will be a focus of the section of the cylinder.

(4) The focal ellipse of an ellipsoid corresponds on the flat confocal ellipsoid to the principal section in its plane, and the focus of the principal section corresponds to the umbilic.

(5) If PG be a normal to an ellipsoid, G the foot of the normal on the plane of the focal ellipse, P' the point of the flat confocal, bounded by the focal ellipse, which corresponds to P, G' the point on the ellipsoid corresponding to G on the flat confocal, $P'G'$ will be perpendicular to the plane of the focal ellipse and be equal to PG.

(6) Any point in the plane of the focal ellipse of an ellipsoid will be the focus of two plane sections perpendicular to that plane, which will be real only when the point lies within the trace on that plane and without the focal curve.

(7) The square of the distance between a focus and the corresponding directrix of the section of an ellipsoid, made by the plane of contact with any enveloping cone of revolution, is $\frac{b^6}{(a^2 - b^2)(b^2 - c^2)}$.

(8) If a sphere intersect an ellipsoid in two plane curves, the sphere and ellipsoid will have two common enveloping cones, whose vertices lie on opposite branches of the umbilical focal curve.

(9) The foci of a series of parallel sections of an ellipsoid perpendicular to the plane of a focal curve will lie on an ellipse which touches the trace on that plane and the focal curve.

(10) Every sphere inscribed in a cone of revolution circumscribing an ellipsoid will cut the ellipsoid in plane curves.

(11) If a section of an ellipsoid be taken passing through a focus S, and the corresponding directrix, and if S' be the point on the trace of the

surface such that the eccentric angles of S, S' in the focal curve and the trace respectively are equal; D, D' the extremities of the diameters conjugate to these points, the eccentricity of the section will be $\frac{\alpha\beta}{ab} \cdot \frac{OD'}{OD}$, O being the centre, α, β, the semi-axes of the focal curve, and a, b, of the trace of the surface.

(12) The locus of a point where a tangent plane to a conicoid is intersected by a bifocal line, to which it is perpendicular, is a sphere, unless the conicoid be a paraboloid, in which case it is a plane touching the paraboloid at the vertex.

(13) If a series of confocal paraboloids be touched by parallel planes, the points of contact will all lie in a bifocal line.

(14) If e, e' be the eccentricities of the principal sections (a, b) and (a, c) of an ellipsoid, shew that the distance of two points S, S' on the focal conics in these planes, whose distances from the section (b, c) are x', x'', will be $\frac{e'}{e} x'' \sim \frac{e}{e'} x'$, and that the shortest distance of the corresponding directrices will vary as SS'.

(15) If two conicoids have a common focus S, and a common directrix, and if a tangent to one of the surfaces at P meet the other surface in Q, Q', and the directrix in R, SP will bisect the angle QSQ'.

(16) If from a point upon a focal line of a cone, perpendiculars be let fall upon the tangent planes to this cone, their feet will be upon a circle, the plane of which will be perpendicular to the other focal line.

(17) In every hyperboloid of one sheet two circular cylinders can be inscribed.

(18) In any hyperboloid there are two diameters, such that any two conjugate planes passing through either of them are at right angles, and these diameters are the focal lines of the asymptotic cone of the hyperboloid.

(19) Two tangent planes to a cone intersect the two cyclic planes in four straight lines which are sides of the same cone of revolution, whose axis is perpendicular to the plane of the two sides of contact.

(20) A spherical triangle has a given area and two sides on two fixed circles, prove that its base touches a sphero-conic, and is bisected by the point of contact.

(21) Shew that the equation of the cone containing the locus of the foot of the perpendicular from a focus of a sphero-conic upon a tangent is $(a^2 + c^2)\, x^2 + (b^2 + c^2)\, y^2 = a^2\, (x^2 + y^2 + z^2)$; and that its cyclic sections are the same as those of the cone containing the locus of the intersection of perpendicular tangents.

(22) Two fixed tangents to a sphero-conic are intersected by any third tangent; shew that the arcs joining the focus and the two points of intersection include a constant angle. Shew also that this angle will be a right angle if the fixed tangents intersect on the directrix arc of the sphero-conic.

State the reciprocal theorem.

(23) If the arc joining two points of a sphero-conic pass through a focus, the sum of the cotangents of the arcs between the focus and the two points will be constant.

State the reciprocal theorem.

CHAPTER XVI.

DISCUSSION OF THE GENERAL EQUATION OF THE SECOND DEGREE.

361. Our object in this chapter is to investigate the position of the origin, and the directions of the axes (which we shall suppose to be a rectangular system) by transformation to which any proposed equation of the second degree will assume its simplest form; and also to find the relations among the coefficients of the general equation which discriminate the various kinds of surfaces capable of being represented by the equation.

362. The general equation of the second degree will be written

$$f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy$$
$$+ 2a''x + 2b''y + 2c''z + d = 0,$$

or $u \equiv u_2 + u_1 + d = 0$;

this equation will sometimes be made homogeneous by the introduction of w for the unit-length, which will enable us to employ the known properties of homogeneous functions; in this case we shall have

$$u \equiv ax^2 + by^2 + cz^2 + dw^2 + 2a'yz + 2b'zx + 2c'xy + 2a''xw$$
$$+ 2b''yw + 2c''zw.$$

363. It will be convenient to denote the discriminant of u_2 by $H(u_2)$ or Δ, and that of u in the homogeneous form by $H(u)$; also the minors of $\Delta \equiv \begin{vmatrix} a, & c', & b' \\ c', & b, & a' \\ b', & a', & c \end{vmatrix}$, viz. $bc - a'^2$, $b'c' - aa'\ldots$, by A, A', B, B', C, C'.

It is easily shewn that $B'C' - AA' = a'\Delta$, and $BC - A'^2 = a\Delta$; it follows therefore that when $\Delta = 0$, the minors are connected by the equations

$$B'C' = AA', \quad C'A' = BB', \quad A'B' = CC',$$

$$\text{and } BC = A'^2, \quad CA = B'^2, \quad AB = C'^2,$$

so that A, B, C are all of the same sign.

364. *The discriminant of u can be expressed in the form of a square, when that of u_2 vanishes.*

$$H(u) \equiv \begin{vmatrix} a, & c', & b', & a'' \\ c', & b, & a', & b'' \\ b', & a', & c, & c'' \\ a'', & b'', & c'', & d \end{vmatrix} \begin{matrix} -a''(a''A + b''C' + c''B') \\ = -b''(a''C' + b''B + c''A') \\ -c''(a''B' + b''A' + c''C), \end{matrix}$$

$$\text{but } \frac{C'}{A} = \frac{B}{C'} = \frac{A'}{B'} \text{ and } \frac{B'}{A} = \frac{A'}{C'} = \frac{C}{B'},$$

$$\therefore \frac{a''A + b''C' + c''B'}{A} = \frac{a''C' + b''B + c''A'}{C'} = \frac{a''B' + b''A' + c''C}{B'}$$

$$= \frac{-H(u)}{a''A + b''C' + c''B'};$$

$$\therefore H(u) = -\frac{1}{A}(a''A + b''C' + c''B')^2$$

$$= -\{a''\sqrt{(A)} + b''\sqrt{(B)} + c''\sqrt{(C)}\}^2,$$

where the signs of $\sqrt{(A)}$, $\sqrt{(B)}$, and $\sqrt{(C)}$ must be all the same as, or all different from, those of A, C', and B'.

365. *To find the centre or the locus of centres of a surface of the second degree.*

Let (ξ, η, ζ) be a centre of the locus of $u = 0$, and let the origin be transformed to this point; the transformed equation is $f(x + \xi, y + \eta, z + \zeta) = 0$, and, since the new origin is the point of bisection of all chords drawn through it, each of the coefficients of x, y, z must vanish;

$$\therefore a\xi + c'\eta + b'\zeta + a'' = 0,$$

$$c'\xi + b\eta + a'\zeta + b'' = 0, \qquad (1)$$

$$b'\xi + a'\eta + c\zeta + c'' = 0,$$

Considering ξ, η, ζ as current coordinates, these three equations represent three planes in each of which the centre lies.

The three planes generally intersect in one point (I), but they may have one line common to them all (II), or they may all three coincide (III).

I. In the first case, there will be one centre which may be at a finite (i) or an infinite distance (ii).

i. If the centre be at a finite distance, its coordinates will be given by

$$\begin{vmatrix} a, & c', & b' \\ c', & b, & a' \\ b', & a', & c \end{vmatrix} \times \xi + \begin{vmatrix} a'', & c', & b' \\ b'', & b, & a' \\ c'', & a', & c \end{vmatrix} = 0,$$

and two similar equations.

ii. The centre will be at an infinite distance if any of its coordinates be infinite; thus if ξ be infinite,

$$\Delta \equiv abc - aa'^2 - bb'^2 - cc'^2 + 2a'b'c' = 0,$$

and $a''A + b''C' + c''B'$ must be finite; and we may notice that η cannot at the same time be finite, unless $C' : A = 0$, (Art. 364).

II. In the second case there will be a line of centres, which may be at a finite (i), or an infinite distance (ii).

i. The coordinates of the centre must be indeterminate, for which we have the conditions that $\Delta = 0$ and that the three expressions $a''A + b''C' + c''B'$, $a''C' + b''B + c''A'$, and $a''B' + b''A' + c''C$ vanish, or

$$\begin{vmatrix} a'', & a, & c', & b' \\ b'', & c', & b, & a' \\ c'', & b', & a', & c \end{vmatrix} = 0.$$

If A', B', C' be finite, $\dfrac{a''}{A'} + \dfrac{b''}{B'} + \dfrac{c''}{C'} = 0.$

ii. The line of centres will be at an infinite distance,

(1) If the three planes be parallel, and not more than two of them coincident; the conditions for this are

$$\frac{a}{c'} = \frac{c'}{b} = \frac{b'}{a'} \text{ and } \frac{c'}{b'} = \frac{b}{a'} = \frac{a'}{c},$$

and that $a'a''$, $b'b''$, $c'c''$ shall not be all equal, hence in this case A', B' and C' all vanish.

(2) If one plane be at an infinite distance, and the other two be parallel or coincident; in this case, if the first plane be that at an infinite distance, a, c', b' must all vanish and a'' be finite, also $\frac{b}{a'} = \frac{a'}{c}$ and these must not be equal to $\frac{b''}{c''}$ if the two planes be parallel, but will be equal to $\frac{b''}{c''}$ if they be coincident.

(3) If one be indeterminate and the others parallel but not coincident; suppose the first to be that which is indeterminate, a, c', b', and a'' must all vanish, and $a'c''$, cb'' must be unequal.

(4) If two be at an infinite distance, or if one be at an infinite distance and a second be indeterminate; in this case all the quantities a, b, c, a', b', c' vanish except one of the first three; if c be finite, a'' and b'' will be either one or both finite.

Hence, for every case of (ii), A', B', and C' all vanish.

III. In the third case there will be a plane of centres, which may be at an infinite distance.

In order that the three planes may coincide, we must have

$$\frac{a}{c'} = \frac{c'}{b} = \frac{b'}{a'} = \frac{a''}{b''} \text{ and } \frac{c'}{b'} = \frac{b}{a'} = \frac{a'}{c} = \frac{b''}{c''};$$

therefore all the minors vanish, and $a'a'' = b'b'' = c'c''$.

If the plane be at an infinite distance, all the coefficients of u_2 must vanish, while one at least of a'', b'', c'' is finite.

366. We have shewn that when Δ or $H(u_2)$ is finite, the terms included in u_1 may be removed by transformation, without altering the directions of the axes, but that for every departure from the general case, in which there is a single centre at a finite distance, one of the conditions is that Δ shall vanish, and this condition is independent of all coefficients of u except those of terms of the second degree; it is also the condition that the part u_2 containing the terms of the second degree shall be the product of two factors, real or imaginary, see Art. 88. So that, in every case except where there is a single centre at a finite distance, by choosing coordinate planes, two of which bisect the

angles between the planes $u_2 = 0$, the general equation can be reduced to the form $\beta y^2 + \gamma z^2 + 2\alpha'' x + 2\beta'' y + 2\gamma'' z + \delta = 0$.

This is further reducible to $\beta y^2 + \gamma z^2 + 2\alpha'' x + \delta = 0$, by moving the origin in the plane of yz; and if α'' be not zero, this finally reduces to $\beta y^2 + \gamma z^2 + 2\alpha'' x = 0$, or to $\beta y^2 + \gamma z^2 + \delta = 0$ if $\alpha'' = 0$. If the two factors of u_2 be equal, the equation is reducible to $\gamma z^2 + 2\alpha'' x = 0$ or $\gamma z^2 + \delta = 0$.

367. The loci of equations of the second degree may therefore be classified according to the nature of their centres.

I. Single Centre.

i. At a finite distance.

Ellipsoid.
Hyperboloids of one and two sheets.
Cone, real.
Cone, imaginary (or point-ellipsoid).

ii. At an infinite distance.

Paraboloids, elliptic and hyperbolic.

II. Line of Centres.

i. At a finite distance.

Cylinders, elliptic and hyperbolic.
Line cylinder (limit of ellip. cylinder).
Two planes, intersecting (limit of hyperb. cylinder).

ii. At an infinite distance.

Cylinder, parabolic.

III. Plane of Centres.

i. At a finite distance.

Two planes, parallel (limit of parab. cylinder).

ii. At an infinite distance.

Two planes, one at an infinite distance.
Two planes, both at an infinite distance.

368. We have now to shew that it is always possible to choose such directions of the axes, that the transformed equation shall contain no terms involving yz, zx, and xy, the axes being in both cases supposed to be rectangular.

369. Since our objects in this chapter are, either to determine what kinds of surfaces can be the loci of the general equation; or, given a particular equation, to identify the surface which is its locus, we may avoid complications by considering that if only one of the rectangles, say xy, appear in the equation, we can by rotation of the axes of x and y make this term disappear, so that the equation will be reduced to the form

$$\alpha x^2 + \beta y^2 + \gamma z^2 + 2\alpha'' x + 2\beta'' y + 2\gamma'' z + \delta = 0,$$

and the nature and position of the locus will be at once determined.

In dealing with the general case we shall not therefore always examine the particular modification of the formulæ which would be required if two of the three quantities a', b' and c' were to vanish.

370. *To shew that u_2 can always be reduced to the form $\alpha x^2 + \beta y^2 + \gamma z^2$ by transformation of coordinates, α, β and γ being real quantities.*

The quadric $h(x^2 + y^2 + z^2) - u_2$ will become the product of two linear factors, real or imaginary, if h satisfy the equation

$$(h-a)(h-b)(h-c)$$
$$- a'^2(h-a) - b'^2(h-b) - c'^2(h-c) - 2a'b'c' = 0 \text{ (Art. 88)}.$$

Since the equation is a cubic, one of the values of h must be real, and for this value $h(x^2 + y^2 + z^2) - u_2 = 0$ is the equation of two planes which, whether real or imaginary, have a real line of intersection.

Let this line be the axis of z in a new system of coordinates, so that $h(x^2 + y^2 + z^2) - u_2$ becomes $Ax^2 + 2Bxy + Cy^2$ on transformation, and the term xy may be made to disappear by simple rotation of the axes of x and y.*

Hence, referred to these new axes, u_2 would be reduced to $\alpha x^2 + \beta y^2 + \gamma z^2$, in which α, β, γ are real, although any one or two may vanish, the corresponding cubic being

$$(h-\alpha)(h-\beta)(h-\gamma) = 0.$$

* This method was adopted by Archibald Smith in his Notes on the "Undulatory Theory of Light."—*Camb. Math. Jour.*, vol. I., p. 5.

371. The cubic given in the last article is called the *discriminating cubic*, the coefficients of which, as we have seen in Art. 152, are invariants.

Since the last term is $-\Delta$, it follows that whenever all the roots of the discriminating cubic are different from zero, the locus of the general equation is a central surface.

372. *To separate the roots of the discriminating cubic.*

The discriminating cubic is

$$\phi(h) \equiv (h-a)(h-b)(h-c) - a'^2(h-a) - \ldots = 0,$$

and $-\phi(h)$ is the value of Δ when $a-h$, $b-h$, $c-h$ are written for a, b, c respectively; hence, since $a'\Delta = B'C' - AA'$, we obtain by this substitution

$$a'\phi(h) \equiv (a'h + A')\{(h-b)(h-c) - a'^2\} - (b'h + B')(c'h + C');$$

and, if we write λ, μ, ν for $-\dfrac{A'}{a'}, -\dfrac{B'}{b'}, -\dfrac{C'}{c'}$,

$$\phi(h) \equiv (h-\lambda)\{(h-b)(h-c) - a'^2\} - \frac{a'b'c'}{a'^2}(h-\mu)(h-\nu); \quad (1)$$

therefore $\phi(\lambda)$ has the same sign as $-a'b'c'(\lambda-\mu)(\lambda-\nu)$; hence if λ, μ, ν be in order of magnitude $\phi(\lambda), \phi(\mu), \phi(\nu)$ are $+ - +$, or $- + -$; therefore λ, μ, ν, or

$$a - \frac{b'c'}{a'}, \quad b - \frac{c'a'}{b'}, \text{ and } c - \frac{a'b'}{c'},$$

separate the roots.

If one of the quantities, as a', vanish,

$$\phi(h) = (h-a)(h-b)(h-c) - b'^2(h-b) - c'^2(h-c);$$

therefore $\phi(\infty), \phi(b), \phi(c), \phi(-\infty)$ are $+ - + -$ supposing $b > c$, and the roots will be separated by b and c.

Cauchy's method of separating the roots is given in Todhunter's *Theory of Equations*, in the chapter on Cubics.

373. *To find the conditions that the discriminating cubic may have equal roots.*

In the case of two equal roots, suppose $\beta = \gamma$, then u_2 can be derived by transformation from

$$\alpha x^2 + \beta(y^2 + z^2) \text{ or } (\alpha - \beta)x^2 + \beta(x^2 + y^2 + z^2);$$

$$\therefore\; u_2 \equiv (\alpha - \beta)(lx + my + nz)^2 + \beta(x^2 + y^2 + z^2);$$

$$\therefore\; a = (\alpha - \beta)\, l^2 + \beta, \quad a' = (\alpha - \beta)\, mn,$$

$$b = (\alpha - \beta)\, m^2 + \beta, \quad b' = (\alpha - \beta)\, nl,$$

$$c = (\alpha - \beta)\, n^2 + \beta, \quad c' = (\alpha - \beta)\, lm,$$

$$\frac{b'c'}{a'} = (\alpha - \beta)\, l^2 = a - \beta;$$

$$\therefore\; \beta = a - \frac{b'c'}{a'} = b - \frac{c'a'}{b'} = c - \frac{a'b'}{c'}.$$

These are obviously the conditions that the conicoid may be one of revolution.

If all three roots be equal, u_2 must have been $\alpha(x^2 + y^2 + z^2)$ before transformation; therefore $a = b = c$ and $a' = b' = c' = 0$.

374. Another form of the conditions for two equal roots may be obtained; for $(\beta - a)(\beta - c) = b'^2$ and $(\beta - a)(\beta - b) = c'^2$;

$$\therefore\; (\beta - a)\,(b - c) = b'^2 - c'^2;$$

$$\therefore\; \beta = a + \frac{b'^2 - c'^2}{b - c} = b + \frac{c'^2 - a'^2}{c - a} = c + \frac{a'^2 - b'^2}{a - b};$$

and we may observe that, if $a' = 0$, b' or $c' = 0$, and if a', c' be the two which vanish, $\beta = b$.

375. *To find the equations of the coordinate axes which make the terms in u_2 involving yz, zx, xy disappear.*

When u_2 has been reduced by transformation to $\alpha x^2 + \beta y^2 + \gamma z^2$, one of the new axes is the intersection of the two planes whose equation is, referred to the original axes,

$$u_2 - h(x^2 + y^2 + z^2) = 0, \quad \text{where } h = \alpha,\ \beta \text{ or } \gamma;$$

therefore, by Art. 89, the equations of the axes are found by writing α, β, γ successively for h in

$$x\{b'c' - (a - h)\, a'\} = y\{c'a' - (b - h)\, b'\} = z\{a'b' - (c - h)\, c'\}.$$

These equations do not give the position of the axes directly, if two of the three quantities a', b', c' vanish, but, if a', b' be the two which vanish, it is obvious, from the original equation, that the axis of z will be in the direction of one of the axes.

376. The direction-cosines of the axes can be symmetrically expressed in terms of the roots of the cubic $\phi(h)=0$.

For $(a'\alpha+A')\{(\alpha-b)(\alpha-c)-a'^2\}=(b'\alpha+B')(c'\alpha+C')$ (Art. 372); therefore $(a'\alpha+A')^2\{(\alpha-b)(\alpha-c)-a'^2\}$ is a symmetrical function of the coefficients, hence, if l, m, n be the direction-cosines of the new axis of x,

$$\frac{l^2}{(\alpha-b)(\alpha-c)-a'^2}=\frac{m^2}{(\alpha-c)(\alpha-a)-b'^2}=\frac{n^2}{(\alpha-a)(\alpha-b)-c'^2}$$
$$=\frac{1}{\phi'(\alpha)}=\frac{1}{(\alpha-\beta)(\alpha-\gamma)}.$$

We give also below the method of determining the directions of the axes by means of the definition of a principal plane.

377. *To find the equation of the locus of middle points of a system of parallel chords of a conicoid determined by the general equation.*

Let the equation of the conicoid be $f(x, y, z)=0$, and let (λ, μ, ν) be the direction of the chords to be bisected, (ξ, η, ζ) the middle point of any chord.

Then the equation $f(\xi+\lambda r, \eta+\mu r, \zeta+\nu r)=0$ must have its roots equal and of opposite signs.

This gives the condition

$$\lambda\frac{df}{d\xi}+\mu\frac{df}{d\eta}+\nu\frac{df}{d\zeta}=0,$$

or $(a\xi+c'\eta+b'\zeta)\lambda+(c'\xi+b\eta+a'\zeta)\mu+(b'\xi+a'\eta+c\zeta)\nu=0$,

which is the equation of the diametral plane.

378. *To determine the principal planes of any conicoid.*

A principal plane being perpendicular to the chords which it bisects, we shall have the direction-cosines given by the three equations

$$\begin{aligned} a\lambda+c'\mu+b'\nu&=s\lambda,\\ c'\lambda+b\mu+a'\nu&=s\mu,\\ b'\lambda+a'\mu+c\nu&=s\nu, \end{aligned}\qquad(1)$$

where s is a constant given by the cubic

$(s-a)(s-b)(s-c)-a'^2(s-a)-b'^2(s-b)-c'^2(s-c)-2a'b'c'=0$,

the discriminating cubic which has been already discussed. Since to each of the three values of s there corresponds one system of values of $\lambda : \mu : \nu$, there are, in general, three and only three principal planes.

If, as in the case of a surface of revolution, there are an infinite number of values of $\lambda : \mu : \nu$, we obtain from equations (1)

$$\frac{a-s}{c'} = \frac{c'}{b-s} = \frac{b'}{a'}, \text{ and } \frac{c'}{b'} = \frac{b-s}{a'} = \frac{a'}{c-s};$$

$$\therefore\ s = a - \frac{b'c'}{a'} = b - \frac{c'a'}{b'} = c - \frac{a'b'}{c'}, \text{ as in Art. 373.}$$

379. *To shew that the three principal planes of any conicoid are mutually at right angles.*

Let s_1, s_2, s_3 be the three roots of the discriminating cubic, and let the corresponding values of λ, μ, ν be denoted by the same suffixes; we shall then have

$$\begin{aligned} a\lambda_1 + c'\mu_1 + b'\nu_1 &= s_1\lambda_1, \\ c'\lambda_1 + b\mu_1 + a'\nu_1 &= s_1\mu_1, \\ b'\lambda_1 + a'\mu_1 + c\nu_1 &= s_1\nu_1. \end{aligned} \qquad (1)$$

Multiplying by λ_2, μ_2, ν_2 and adding, we obtain

$$(a\lambda_2 + c'\mu_2 + b'\nu_2)\,\lambda_1 + (c'\lambda_2 + b\mu_2 + a'\nu_2)\,\mu_1 + (b'\lambda_2 + a'\mu_2 + c\nu_2)\,\nu_1$$
$$= s_1\,(\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2);$$

$$\therefore\ s_2\lambda_2 \cdot \lambda_1 + s_2\mu_2 \cdot \mu_1 + s_2\nu_2 \cdot \nu_1 = s_1\,(\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2),$$

$$\text{whence } (s_2 - s_1)\,(\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2) = 0.$$

Hence, if two roots of the cubic be unequal the corresponding principal planes will be at right angles.

We may make use of equations (1) to shew that the equation of the surface referred to the principal planes as coordinate planes will be of the form $\alpha x^2 + \beta y^2 + \gamma z^2 + \delta = 0$, in which α, β, γ are the roots of the discriminating cubic, for, on transformation, the coefficient of x^2 will be

$$a\lambda_1^2 + b\mu_1^2 + c\nu_1^2 + 2a'\mu_1\nu_1 + 2b'\nu_1\lambda_1 + 2c'\lambda_1\mu_1$$
$$= s_1\,(\lambda_1^2 + \mu_1^2 + \nu_1^2) = s_1, \text{ by (1)},$$

and similarly, $\beta = s_2,\ \gamma = s_3$.

380. *To distinguish the surfaces represented by an equation for which the roots of the discriminating cubic are finite.*

In this case there is a centre at a finite distance, to which if the origin be transferred, the direction of a new system of axes can be chosen (Art. 370), such that

$$u \equiv ax^2 + by^2 + cz^2 + dw^2$$
$$+ 2a'yz + 2b'zx + 2c'xy + 2a''xw + 2b''yw + 2c''zw = 0,$$

will become by transformation $\alpha x^2 + \beta y^2 + \gamma z^2 + \delta w^2 = 0$, w being written for the unit.

The transformation will be effected by substituting $lx + my + nz + \xi w$ for x, and similar expressions for y and z, w being unchanged; the discriminants, being invariants, are therefore equal,* since the modulus of transformation

$$= \begin{vmatrix} l, & m, & n, & 0 \\ l', & m', & n', & 0 \\ l'', & m'', & n'', & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix} = 1;$$

$$\therefore\ H(u) = \begin{vmatrix} a, & c', & b', & a'' \\ c', & b, & a', & b'' \\ b', & a', & c, & c'' \\ a'', & b'', & c'', & d \end{vmatrix} = \begin{vmatrix} \alpha, & 0, & 0, & 0 \\ 0, & \beta, & 0, & 0 \\ 0, & 0, & \gamma, & 0 \\ 0, & 0, & 0, & \delta \end{vmatrix} = \alpha\beta\gamma\delta;$$

$$\therefore\ \alpha x^2 + \beta y^2 + \gamma z^2 = -\frac{H(u)}{\alpha\beta\gamma}.$$

Hence, we have the following table for the case in which $\alpha\beta\gamma$ or Δ is finite, and $\alpha > \beta > \gamma$, by which it may be seen how the loci are distinguished:

α	β	γ	$H(u)$	
+	+	+	−	Ellipsoid
+	+	−	+	Hyperboloid, one sheet
+	−	−	−	Hyperboloid, two sheets
+ +	+ −	− −	0	Cone, real
+	+	+	0	Cone, imaginary, or point
+	+	+	+	Imaginary locus

* Salmon's *Higher Algebra*, Arts. 118 and 23.

In order that α, β, and γ may be all positive, $a+b+c$ and Δ must be positive. If the locus be a point, or rather an indefinitely small ellipsoid, the section of $u_2=0$ by each coordinate plane must be a point-ellipse; therefore each of the quantities $bc-a'^2$, $ca-b'^2$, and $ab-c'^2$ must be positive.

The conditions for surfaces of revolution are obtained in Art. 373.

381. *To distinguish the surfaces represented by an equation, for which one of the roots of the cubic vanishes, and the centre is single and at an infinite distance.*

The conditions that the centre may be at an infinite distance are that $\Delta=0$, and that one or more of the three quantities below shall be finite,

$$a''A+b''C'+c''B',$$
$$a''C'+b''B+c''A',$$
$$a''B'+b''A'+c''C.$$

The surfaces will be the elliptic or hyperbolic paraboloid, according as the roots of $h^2-(a+b+c)h+A+B+C=0$, have the same or opposite signs, *i.e.* as $A+B+C$ is + or −; but, by Art. 363, A, B, C have the same sign, hence

A, B, and $C+$ gives an elliptic paraboloid,

A, B, and $C-$,, hyperbolic paraboloid.

382. *To distinguish the surfaces represented by the general equation when there is a line of centres at a finite distance.*

The conditions that there may be a line of centres are $\Delta=0$ and $H(u)=-\{a''\sqrt{(A)}+b''\sqrt{(B)}+c''\sqrt{(C)}\}^2=0$; or, if A', B', C' be finite $\dfrac{a''}{A'}+\dfrac{b''}{B'}+\dfrac{c''}{C'}=0$. The equations of the line of centres are $A'\xi-a'a''=B'\eta-b'b''=C'\zeta-c'c''=\rho$ suppose; therefore, if we transfer the origin to any point in the line of centres defined by some value of ρ, the equation of the surface will become

$$u_2+a''\xi+b''\eta+c''\zeta+d=0,$$

$$\text{or } u_2+\frac{a'a''^2}{A'}+\frac{b'b''^2}{B'}+\frac{c'c''^2}{C'}+d=0,$$

since the coefficient of ρ vanishes.

If a', b', c' be all finite, A', B', C' will be so also; but if a', for instance, vanish, the other two being finite, A' will be finite, and, by Art. 363, if $B'=0$, then $A=0$, and $C'=0$, and b and c vanish; hence, recurring to the original equations for determining the centre, we easily obtain the equation $u_2 - \frac{2a''b''}{c'} + \frac{ab''^2}{c'^2} + d = 0$, and the condition $b'b'' = c'c''$.

If two roots of the discriminating cubic be finite, since u_2 is reducible to the form $\beta y^2 + \gamma z^2$, the surfaces represented by the equation will be in the general case in which

$$\frac{a'a''^2}{A'} + \frac{b'b''^2}{B'} + \frac{c'c''^2}{C'} + d \text{ is finite,}$$

A, B, and $C+$, an elliptic cylinder,

A, B, and $C-$, a hyperbolic cylinder;

$$\text{when } \frac{a'a''^2}{A'} + \ldots = 0,$$

A, B, and $C+$, a line cylinder,

A, B, and $C-$, two intersecting planes.

If only one root be finite, u_2 is reducible to γz^2, but in this case, since $A+B+C=0$, A, B, C, being all of the same sign, must vanish separately, from which it follows that A', B', C' also vanish, and there cannot be a line of centres at a finite distance.

383. *To distinguish the surfaces when there is a line of centres at an infinite distance.*

In this case $A'=B'=C'=0$; therefore $A=B=C=0$; two of the roots of the discriminating cubic must therefore vanish; also $a'a''$, $b'b''$, $c'c''$ must not be all equal.

Since $aa'=b'c'$, &c., u_2 can be put into the form

$$a'b'c'\left(\frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'}\right)^2,$$

and the only surface represented is a parabolic cylinder.

384. *To distinguish the surfaces for which there is a plane of centres.*

In this case, as in the last, the minors all vanish, and we have in addition $a'a''=b'b''=c'c''$; the equation may therefore be written

$$a'b'c'\left(\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'}\right)^2+2a'a''\left(\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'}\right)+d=0.$$

The surface represented consists of two parallel planes unless $a'b'c'd=a''^2a'''^2$, or $d=\frac{a'''^2}{a}=\frac{b'''^2}{b}=\frac{c'''^2}{c}$, in which case they are coincident.

One of the planes will be at an infinite distance if a, b, c, a', b', c' all vanish while one at least of the other quantities remains finite.

385. The results in the general case may be tabulated as follows, if v be written for $\frac{a'a'''^2}{A'}+\frac{b'b'''^2}{B'}+\frac{c'c'''^2}{C'}+d$, and v' for $(a'a''-b'b'')^2+(b'b''-c'c'')^2$, where f denotes 'finite' and α, β, γ are the roots of the discriminating cubic, $\alpha>\beta>\gamma$.

Δ	α	β	γ	$H(u)$	A, B, C	v	v'	
+	+	+	+	−				Ellipsoid
−	+	+	−	+				Hyperboloid, one sheet
+	+	−	−	−				Hyperboloid, two sheets
− +	+ +	+ −	− −	0				Cone, real
+	+	+	+	0	+			Cone, imaginary, or point-ellipsoid
0	0	+	+	f	+			Paraboloid, elliptic
0	0	+	−	f	−			Paraboloid, hyperbolic
0	0	+	+	0	+	f		Cylinder, elliptic
0	0	+	+	0	+	0		Cylinder, line
0	0	+	−	0	−	f		Cylinder, hyperbolic
0	0	+	−	0	−	0		Planes, intersecting
0	0	0	+	0	0		f	Cylinder, parabolic
0	0	0	+	0	0		0	Planes, parallel

For coincident planes $d=\frac{a'''^2}{a}=\frac{b'''^2}{b}=\frac{c'''^2}{c}$.

For two planes, one at an infinite distance, $a, b, c, a', b', c'=0$, one at least of a'', b'', c'' finite.

For two planes at an infinite distance, d alone finite.

386. *Processes for finding the locus of any given equation.* When a particular equation of the second degree is presented to us, in order to discover what species of surface it represents, we would recommend the student first to form the discriminating cubic, and it will then be seen whether the last term Δ vanishes or not.

I. If Δ be different from zero, we must find the centre, transfer the origin to it, and by changing the directions of the axes reduce the equation to the form $\alpha x^2+\beta y^2+\gamma z^2+\delta=0$, where α, β, γ are the roots of the discriminating cubic, which can always be found approximately, at all events their signs can be determined by Des Cartes' rule; and δ has been shewn to be $\dfrac{H(u)}{\Delta}$, or in particular cases may be found more easily without the use of the determinants.

II. If $\Delta=0$, and $A+B+C$ be not zero, in which case the two roots β, γ will be finite, it will be best to determine the directions of the axes which correspond to the three roots $0, \beta, \gamma$, and to suppose the origin so chosen that the equation becomes either

$$\beta y^2+\gamma z^2+2\alpha'' x=0, \tag{1}$$

$$\text{or } \beta y^2+\gamma z^2+\delta=0. \tag{2}$$

If we do not require the position of the vertex, the value of α'' in (1) can be found by equating the discriminants, by which we obtain $\beta\gamma\alpha''^2=-\dfrac{1}{A}(a''A+b''C'+c''B')^2$, and if we find that α'' vanishes, we take case (2).

But if in case (1) the coordinates ξ, η, ζ of the vertex be required, we think that the best method is to transfer to this point, and observe that the result must be

$$u_2+2\alpha''(lx+my+nz)=0,$$

if (l, m, n) be the direction of the new axis of x; so that we obtain the equations for determining ξ, η, ζ and α'',

$$a\xi + c'\eta + b'\zeta + a'' = l\alpha'',$$
$$c'\xi + b\eta + a'\zeta + b'' = m\alpha'',$$
$$b'\xi + a'\eta + c\zeta + c'' = n\alpha'',$$
$$a''\xi + b''\eta + c''\zeta + d + (l\xi + m\eta + n\zeta)\,\alpha'' = 0.$$

From the first three equations

$$a''A + b''C' + c''B' = (lA + mC' + nB')\,\alpha'',$$

which gives α'' without ambiguity, and the fourth equation with two of the former determines ξ, η, and ζ.

If $\alpha'' = 0$, we shall have in case (2) three equations, equivalent to two independent equations, which will determine the axis of the surface, and we can obtain δ from a fourth equation combined with two of the former; thus eliminating ξ and η from the last three,

$$\delta\,(a'c' - bb') = b''\,(b'b'' - a'a'') + c''\,(ba'' - c'b'') + d\,(a'c' - bb'),$$

since the coefficient of $\zeta = a''A + b''C' + c''B' = 0$, when $\alpha'' = 0$.

Thus the position of the locus is determined completely in both cases.

III. If $\Delta = 0$ and $A + B + C = 0$, the equation is reducible to one of the forms $\gamma z^2 + 2\alpha'' x = 0$ (1), $\gamma z^2 + \delta = 0$ (2).

In this case, since $a = \dfrac{b'c'}{a'}$ &c., the original equation is easily put into the form

$$a'b'c'\left(\frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'}\right)^2 + 2\,(a''x + b''y + c''z) + d = 0;$$

if (l, m, n), (l', m', n') be the directions of the new axes of x, z, and (ξ, η, ζ) be any point on the line of vertices in (1), the equation referred to that point as origin and to axes parallel to the original axes will be

$$\gamma\,(l'x + m'y + n'z)^2 + 2\alpha''\,(lx + my + nz) = 0,$$

whence $l'a' = m'b' = n'c' = \sqrt{\dfrac{a'b'c'}{\gamma}} = \sqrt{\left(\dfrac{abc}{a+b+c}\right)}$.

We obtain also the equations

$$\gamma l'(l'\xi + m'\eta + n'\zeta) + a'' = l\alpha'',$$

$$\gamma m'(l'\xi + m'\eta + n'\zeta) + b'' = m\alpha'',$$

$$\gamma n'(l'\xi + m'\eta + n'\zeta) + c'' = n\alpha'',$$

$$a''\xi + b''\eta + c''\zeta + (l\xi + m\eta + n\zeta)\alpha'' + d = 0;$$

$$\therefore\ \gamma(l'\xi + m'\eta + n'\zeta) + a''l' + b''m' + c''n' = 0, \qquad (3)$$

and $\alpha''^2 = a''^2 + b''^2 + c''^2 - (a''l' + b''m' + c''n')^2$

$$= (b''n' - c''m')^2 + (c''l' - a''n')^2 + (a''m' - b''l')^2, \qquad (4)$$

also $2(a''\xi + b''\eta + c''\zeta) + \dfrac{1}{\gamma}(a''l' + b''m' + c''n')^2 + d = 0, \qquad (5)$

(4) gives the latus rectum of the principal parabolic section, (3) and (5) are the equations of the line of vertices.

The value of α'' shews the necessity of the condition that $a'a''$, $b'b''$, $c'c''$ must not be all equal.

If $a'a'' = b'b'' = c'c''$, see Art. 384.

387. We shall conclude this chapter by applying some of its processes to two numerical examples, and we shall also shew how the discriminating cubic may be sometimes employed with advantage when we cannot find the roots, by investigating directly the construction for the axes of a cone enveloping a conicoid, and the locus of its vertex when it is a right cone.

388. *To find the surface whose equation is*

$$32x^2 + y^2 + z^2 + 6yz - 16zx - 16xy - 6x - 12y - 12z + 18 = 0.$$

The discriminating cubic is

$$(h - 32)(h - 1)^2 - 9(h - 32) - 128(h - 1) - 6.64 = 0,$$

the last term of which is 0 and the roots 0, 36, -2.

The equations of the axes are known from

$$\{64 + 3(h - 32)\}x = \{-24 - 8(h - 1)\}y = \{-24 - 8(h - 1)\}z,$$

which give $\quad 2x = y = z; \quad x = -4y = -4z; \quad x = 0,\ y = z;$

and the direction-cosines are

$$(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}), \ (\tfrac{2}{3}\sqrt{2}, -\tfrac{1}{6}\sqrt{2}, -\tfrac{1}{6}\sqrt{2}), \text{ and } (0, \tfrac{1}{2}\sqrt{2}, -\tfrac{1}{2}\sqrt{2}).$$

The equation being reduced to $36y^2 - 2z^2 + 2\alpha'' x = 0$, by the equation of invariants

$$-\tfrac{1}{8}(3.8 + 6.16 + 6.16)^2 = -36.2\alpha''^2; \quad \therefore \ \alpha'' = \pm 9.$$

If (ξ, η, ζ) be the vertex referred to the original axes, when we transfer to this point the equation must reduce to

$$u_2 + 2\alpha''\left(\tfrac{1}{3}x + \tfrac{2}{3}y + \tfrac{2}{3}z\right) = 0;$$

$$\begin{aligned} \therefore\ 32\xi - 8\eta - 8\zeta - 3 &= \tfrac{1}{3}\alpha'' \\ -\ 8\xi + \eta + 3\zeta - 6 &= \tfrac{2}{3}\alpha'' \\ -\ 8\xi + 3\eta + \zeta - 6 &= \tfrac{2}{3}\alpha''; \end{aligned}$$

$$\therefore\ -3 - 2.6 - 2.6 = 3\alpha''; \quad \therefore\ \alpha'' = -9,$$

and since the constant term vanishes

$$\tfrac{1}{3}\alpha''(\xi + 2\eta + 2\zeta) - 3\xi - 6\eta - 6\zeta + 18 = 0,$$

$$\text{or} \quad \xi + 2\eta + 2\zeta = 3;$$

$$\therefore\ \eta = \zeta = 2\xi = \tfrac{2}{3}.$$

389. *To find the surface whose equation is*

$$x^2 - 2y^2 + 2z^2 + 3zx - xy - 2x + 7y - 5z - 3 = 0.$$

The discriminating cubic is

$$(h-1)(h^2-4) - \tfrac{9}{4}(h+2) - \tfrac{1}{4}(h-2) = 0, \ \text{roots } 0, \ \frac{\pm 3\sqrt{3}+1}{2},$$

$$H(u) = -\frac{1}{\Delta}(Aa'' + C'b'' + B'c'')^2 = 0,$$

and the equation is reducible to

$$\frac{3\sqrt{(3)}+1}{2}y^2 - \frac{3\sqrt{(3)}-1}{2}z^2 + \delta = 0.$$

The equations determining any point (ξ, η, ζ) of the axis of the surface are

$$\begin{aligned} 2\xi - \eta + 3\zeta - 2 &= 0, \\ -\ \xi - 4\eta + 7 &= 0, \qquad (1) \\ 3\xi + 4\zeta - 5 &= 0, \end{aligned}$$

which give the straight line

$$\xi = -4\eta + 7 = \tfrac{1}{3}(-4\zeta + 5); \qquad (2)$$

therefore multiplying the equations (1) by ξ, η, ζ, and adding

$$2\delta = 2f(\xi, \eta, \zeta) = -2\xi + 7\eta - 5\zeta - 6 = 0.$$

The equation therefore represents two intersecting planes whose line of intersection is given by (2).

390. *To find the axes of the conical envelope of a central conicoid.*

The equation of the cone referred to its vertex as origin is

$$\sigma(ax^2 + by^2 + cz^2) = (afx + bgy + chz)^2,$$

σ being written for $af^2 + bg^2 + ch^2 - 1$, the discriminating cubic in this case is

$$\begin{aligned}(s - \sigma a + a^2f^2)&(s - \sigma b + b^2g^2)(s - \sigma c + c^2h^2) \\ &- b^2c^2g^2h^2(s - \sigma a + a^2f^2) - c^2a^2h^2f^2(s - \sigma b + b^2g^2) \\ &- a^2b^2f^2g^2(s - \sigma c + c^2h^2) + 2a^2b^2c^2f^2g^2h^2 = 0,\end{aligned}$$

or writing s_1, s_2, s_3 for $s - \sigma a$, $s - \sigma b$, $s - \sigma c$

$$s_1s_2s_3 + a^2f^2s_2s_3 + b^2g^2s_3s_1 + c^2h^2s_1s_2 = 0;$$

$$\therefore \frac{a^2f^2}{s - \sigma a} + \frac{b^2g^2}{s - \sigma b} + \frac{c^2h^2}{s - \sigma c} + 1 = 0,$$

$$\text{and } af^2 + bg^2 + ch^2 - 1 = \sigma;$$

therefore multiplying the first equation by σ, and adding

$$\frac{saf^2}{s - \sigma a} + \frac{sbg^2}{s - \sigma b} + \frac{sch^2}{s - \sigma c} = 1,$$

$$\text{or } \frac{f^2}{\frac{1}{a} - \frac{\sigma}{s}} + \frac{g^2}{\frac{1}{b} - \frac{\sigma}{s}} + \frac{h^2}{\frac{1}{c} - \frac{\sigma}{s}} = 1.$$

Now by Art. 375 the direction-cosines of the axis corresponding to s are inversely proportional to $a'\left(s - a + \dfrac{b'c'}{a'}\right)$, &c., and in this case $a - \dfrac{b'c'}{a'}$, and a' are replaced by σa and $-bcgh$, hence the direction-cosines of this axis of the cone are in the ratio

$$\frac{f}{\frac{1}{a} - \frac{\sigma}{s}} : \frac{g}{\frac{1}{b} - \frac{\sigma}{s}} : \frac{h}{\frac{1}{c} - \frac{\sigma}{s}};$$

thus we have proved, by the method of this chapter, that the axes of the cone are the normals to the confocals through the vertex.

The conditions of Art. 373, that the conical envelope shall be a surface of revolution, are, since the expression $a - \frac{b'c'}{a'}$ becomes σa, either that $a = b = c$, in which case the enveloped conicoid is a sphere, or that f, g, or $h = 0$; suppose $h = 0$, then one of the three corresponding values of s will be σc;

$$\therefore \frac{f'^2}{\frac{1}{a} - \frac{1}{c}} + \frac{g'^2}{\frac{1}{b} - \frac{1}{c}} = 1,$$

hence the vertex must lie on a focal conic.

XVI.

(1) Find the nature of the surfaces represented by the following equations:

1. $3z^2 - x^2 - y^2 + 4xy = a^2$.
2. $yz + zx + xy = a^2$.
3. $x^2 + y^2 + z^2 + 2xy + 2yz + 4zx = 1$.
4. $x^2 + y^2 + 2(xy + yz + zx) = a^2$.
5. $2z^2 - 5x^2 - 2y^2 + 10xy + 4yz + 4y + 16z + 16 = 0$.
6. $x^2 + 2(yz + zx + xy) + 2(z - y - 1) = 0$.
7. $x^2 + y^2 + 3z^2 + 3yz + zx + xy - 7x - 14y - 25z + d = 0$.
8. $5y^2 - 2x^2 - z^2 - 4xy - 6yz + 8zx = 1$.

Prove the following results:

1. Hyperboloid of one sheet. 2. Hyperboloid of revolution, eccentricity of generating hyperbola $= \sqrt{(\frac{3}{2})}$. 3. Hyperboloid of one sheet, axes $\sqrt{(6)} - \sqrt{(2)}$, $\sqrt{(6)} + \sqrt{(2)}$, 1; direction-cosines of axes in the ratios $\{1, \pm\sqrt{(3)} - 1, 1\}$, $(1, 0, -1)$. 4. Hyperbolic cylinder. 5. Hyperboloid of one sheet, centre $(\frac{6}{5}, \frac{6}{5}, -\frac{26}{5})$. 6. Cone, direction-cosines of axes $\{0, \frac{1}{2}\sqrt{(2)}, -\frac{1}{2}\sqrt{(2)}\}$, $\{\pm\frac{1}{2}\sqrt{(2)}, \frac{1}{2}, \frac{1}{2}\}$. 7. Ellipsoid, point or impossible according as $d <=> 55$. 8. Hyperbolic cylinder.

(2) The equation $7x^2 + 8y^2 + 4z^2 - 7yz - 11zx - 7xy = a^4$ represents an hyperboloid of one sheet whose greater real axis makes with the axis of z an angle $\tan^{-1}\sqrt{(2)}$.

(3) The equation

$$ax^2 + 4y^2 + 9z^2 + 12yz + 6zx + 4xy + 2a'x + 2b'y + 2c'z + d = 0$$

will in general represent an elliptic paraboloid, a parabolic cylinder, or a

hyperbolic paraboloid, according as $a >=< 1$. What surfaces will it represent in the following cases:

i. $3b'' = 2c''$, $a >=< 1$. ii. $6a'' = 3b'' = 2c''$, $a = 1$.

(4) The equation $(cy - bz)^2 + (az - cx)^2 + (bx - ay)^2 = 1$ represents a right circular cylinder, the equations of whose axis are $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

(5) The equation $a(y - z)^2 + b(z - x)^2 + c(x - y)^2 = d^2$ represents a cylinder, which is hyperbolic when $bc + ca + ab$ is negative; and which, when $bc + ca + ab$ is positive, is elliptic or impossible according as $a + b + c$ is positive or negative; if $a + b + c = 0$, the principal section will be a rectangular hyperbola.

(6) The surface represented by the equation

$$a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 1$$

is an hyperboloid of one sheet, and the sum of the squares on its real axes is equal to the square on its conjugate axis.

(7) The equation $ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 0$ will represent a right cone, whose vertical angle is θ, if

$$\frac{aa' - b'c'}{a'} = \frac{bb' - c'a'}{b'} = \frac{cc' - a'b'}{c'} = (a + b + c)\frac{1 + \cos\theta}{1 + 3\cos\theta}.$$

(8) If a cone whose vertex is the origin and base a plane section of the surface $ax^2 + by^2 + cz^2 = 1$ be a cone of revolution, the plane must touch one of the cylinders $(b - a)y^2 + (c - a)z^2 = 1$, $(c - b)z^2 + (a - b)x^2 = 1$, $(a - c)x^2 + (b - c)y^2 = 1$.

(9) Discuss the different surfaces represented by the equation

$$x^2 + (2m^2 + 1)(y^2 + z^2) - 2(yz + zx + xy) = 2m^2 - 3m + 1,$$

as m varies from $-\infty$ to $+\infty$; considering particularly the critical values -1, $\frac{1}{2}$, and 1.

(10) Prove that, when $bb' = c'a'$, and $cc' = a'b'$, the equation

$$ax^2 + \ldots + 2a'yz + \ldots + 2a''x + \ldots + f = 0,$$

represents in general a paraboloid whose axis is parallel to the straight line $x = 0$, $c'y + b'z = 0$.

(11) The section of the surface $yz + zx + xy = a^2$ by the plane $lx + my + nz = p$ will be a parabola if $l^{\frac{1}{2}} + m^{\frac{1}{2}} + n^{\frac{1}{2}} = 0$; and that of the surface

$$x^2 + y^2 + z^2 - 2yz - 2xy = a^2$$

will be a parabola if $mn + nl + lm = 0$.

(12) The radius r of the central circular sections of the surface $ayz + bzx + cxy = 1$ is given by the equation $abcr^6 + (a^2 + b^2 + c^2)r^4 = 4$ and the direction-cosines of the sections by the equations

$$\frac{l(m^2 + n^2)}{a} = \frac{m(n^2 + l^2)}{b} = \frac{n(l^2 + m^2)}{c} = lmnr^2.$$

(13) The semi-axes of a central section of the surface $ayz + bzx + cxy + abc = 0$ made by a plane whose direction-cosines are l, m, n, are given by the equation $r^4 (2bcmn + \ldots - a^2l^2 - \ldots) - 4abcr^2 (amn + \ldots) + 4a^2b^2c^2 = 0$.

(14) Prove that the section of the surface

$$ax^2 + \ldots + 2a'yz + \ldots + 2a''x + \ldots + d = 0,$$

by the plane $lx + my + nz = 0$ will be a rectangular hyperbola, if

$$l^2 (b + c) + m^2 (c + a) + n^2 (a + b) = 2a'mn + 2b'nl + 2c'lm;$$

and a parabola if $l^2 (bc - a'^2) + \ldots + \ldots + 2mn (b'c' - aa') + \ldots + \ldots = 0$. Explain why this last equation becomes identical if $b'c' = aa'$, $c'a' = bb'$, and $a'b' = cc'$.

(15) The area of the section of the conicoid

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$$

through the extremities of its principal axes is

$$\frac{2\pi}{3\sqrt{(3)}} \sqrt{\left(\frac{a + b + c}{abc + 2fgh - af^2 - bg^2 - ch^2}\right)}.$$

(16) A cone is described whose base is a given conic, and one of whose axes passes through a fixed point in the plane of the conic, prove that the locus of the vertex is a circle.

(17) Find the equations of the hyperboloid, three of whose generating lines are $x = 0$, $y = a$; $y = 0$, $z = a$; and $z = 0$, $x = a$; shew that it is a surface of revolution, and find the eccentricity of a meridian section.

(18) If r be any semi-axis of the quadric

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \equiv (a, b, c, f, g, h)(x, y, z)^2 = 1,$$

prove that the values of r will be given by the equation

$$\frac{gh}{gh - af + \frac{f}{r^2}} + \frac{hf}{hf - bg + \frac{g}{r^2}} + \frac{fg}{fg - ch + \frac{h}{r^2}} = 1.$$

(19) Find the locus of the centres of the surfaces represented by the equation $x^2 + y^2 - z^2 + 2pxz + 2qyz - 2ax - 2by + 2cz = 0$, a, b, c being given positive numbers, and p and q variable parameters.

i. When p and q vary in every possible manner.

ii. When they vary in such a manner that the equation may always represent a cone.

Distinguish the respective parts of the locus which correspond to hyperboloids of one and two sheets.

(20) The surface whose equation, referred to axes inclined at angles α, β, γ, is $ax^2 + by^2 + cz^2 = 1$, will be one of revolution if

$$\frac{a \cos\alpha}{\cos\alpha - \cos\beta\cos\gamma} = \frac{b\cos\beta}{\cos\beta - \cos\gamma\cos\alpha} = \frac{c\cos\gamma}{\cos\gamma - \cos\alpha\cos\beta}.$$

(21) The surface whose equation, referred to axes inclined at angles α, β, γ, is $ayz + bzx + cxy = 1$, will be one of revolution if

$$\frac{a}{1 \pm \cos\alpha} = \frac{b}{1 \pm \cos\beta} = \frac{c}{1 \pm \cos\gamma};$$

one or three of the ambiguities being taken negative.

(22) The equation $ax^2 + by^2 + cz^2 = 1$ may, when referred to oblique axes, be transformed into the equation $2m(yz + zx + xy) = 1$ in an infinite number of ways. If a', b', c' be the cosines of the angles between the axes, shew that

$$\frac{m}{a} + \frac{m}{b} + \frac{m}{c} = a' + b' + c' - \tfrac{3}{2},$$

$$m^2\left(\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}\right) = b'c' + c'a' + a'b' - a' - b' - c',$$

$$\text{and } \frac{2m^3}{abc} = 1 - a'^2 - b'^2 - c'^2 + 2a'b'c'.$$

If the oblique axes be mutually inclined at angles of 60°, shew that either $-2a = b = c$, $a = -2b = c$, or $a = b = -2c$.

(23) Shew that the hyperboloid, whose equation, referred to oblique axes inclined at angles $\cos^{-1}a'$, $\cos^{-1}b'$, $\cos^{-1}c'$, is

$$(1 - a')\,yz + (1 - b')\,zx + (1 - c')\,xy = d,$$

is an hyperboloid of revolution, whose equation, referred to its principal axes, is

$$x^2 + y^2 - 2\,\frac{(1 - a')(1 - b')(1 - c')}{1 - a'^2 - b'^2 - c'^2 + 2a'b'c'}\,z^2 + 2d = 0.$$

(24) If the equation of an hyperboloid, referred to oblique axes inclined at angles α, β, γ, such that $\alpha + \beta + \gamma = \pi$, be

$$yz\cos\alpha + zx\cos\beta + xy\cos\gamma = d^2,$$

shew that the length of one of its axes will be $4d$: and that the eccentricity (e) of its principal elliptic sections will be given by the equation

$$\frac{4e^4}{1 - e^2} = \frac{1 - 8\cos\alpha\cos\beta\cos\gamma}{\cos^2\alpha\cos^2\beta\cos^2\gamma}.$$

(25) Three fixed rectangular axes are taken, and a fixed line through the origin whose direction-cosines are λ, μ, ν. If any rigid surface be turned about this line through an angle θ, the equation of such a surface in its new position may be derived from its equation in the old one by changing x into $x\cos\theta + \lambda(\lambda x + \mu y + \nu z)(1 - \cos\theta) + (\mu z - \nu y)\sin\theta$, with similar changes for y and z.

Hence, defining an axis of a conicoid as a diameter, such that by revolution about it through two right angles every point of the surface returns to the surface again, deduce the ordinary cubic equation for the determination of the axes.

CHAPTER XVII.

DEGREES AND CLASSES OF SURFACES. DEGREES OF CURVES AND TORSES. COMPLETE AND PARTIAL INTERSECTIONS OF SURFACES.

391. HAVING already fully investigated the nature of the surfaces represented by the general equation of the second degree, we will proceed to the loci of equations of higher degrees, which we may consider as equations either in three-plane or four-plane coordinates: in the latter case we may suppose the equations homogeneous, without loss of generality.

392. Surfaces which are represented by rational and integral algebraical equations are arranged according to the degrees of these equations when plane coordinates are used, and according to classes when tangential or point coordinates are used.

A surface is of the n^{th} *degree* when the equation of which it is the locus is of the n^{th} degree in the coordinates of any point of the locus; the geometrical equivalent being that a surface is of the n^{th} degree when an arbitrary straight line intersects it in n points, real or imaginary.

A surface is of the n^{th} *class* when n tangent planes, real or imaginary, can be drawn to it through an arbitrary straight line.

If p', q', r', s' and p'', q'', r'', s'' be the point coordinates of two planes, the coordinates of any plane passing through their line of intersection will be $lp' + mp''$, $lq' + mq''$...(Art. 128), $l : m$ being an arbitrary ratio, and the particular planes which touch a surface whose tangential equation is $F(p, q, r, s) = 0$, supposed a homogeneous algebraical equation of the n^{th} degree, will be determined by the values of $l : m$ which satisfy the equation $F(lp' + mp'', \ldots) = 0$; the number of values of the ratio will be n, and this will therefore be the class of the surface.

393. Curves and Torses are arranged according to their degrees.

A curve of the n^{th} degree is one which intersects an arbitrary plane in n points, real or imaginary.

A torse of the n^{th} degree is one to which n tangent planes, real or imaginary, can be drawn through an arbitrary point.

Other classifications of curves and torses will be explained hereafter.

394. Among the various methods of treating of curves which have been proposed, one is to consider them as the intersection of surfaces whose equations are given. In this method the difficulty arises, to which allusion has been made (Art. 13), viz. that extraneous curves may be introduced which are not the subjects of investigation.

If any curve be supposed to be given in space, it is impossible generally to determine two surfaces which shall contain no other points but points which lie on the proposed curve; but among all the surfaces which may be drawn through a curve, it is desirable to obtain the simplest forms of surfaces of which the curve shall be the partial intersection.

395. The number of points in which three surfaces intersect, which are of the m^{th}, n^{th}, and p^{th} degrees respectively, is mnp, unless they intersect in a common curve, in which case it is infinite.

For the proof of this proposition, the student is referred to Salmon's *Treatise on Higher Algebra*, Lesson VIII., on the number of solutions of three equations in three unknown quantities.

The student may be able to satisfy himself of the truth of the proposition, by considering that the number of points in which the surfaces intersect will, by the law of continuity, be unaltered, if we substitute particular instead of the general forms of the surfaces. If the surfaces respectively consist of m, n, p arbitrary planes, it is obvious that the number of their common points of intersection will be mnp, each point being the intersection of three planes, taken one from each system.

396. *The complete intersection of two surfaces of the* m^{th} *and* n^{th} *degrees respectively, is a curve of the* mn^{th} *degree.*

Let a plane intersect the surfaces, the number of points of intersection of the plane with the surfaces is mn, this is therefore the number of points in which the plane cuts the curve, and the curve is of the mn^{th} degree.

397. *To find the number of conditions which a surface of the* n^{th} *degree may be made to satisfy.*

The number of constants in the general equation of the n^{th} degree is evidently the number of homogeneous products of four things of n dimensions, and is therefore

$$= \frac{4.5\ldots(4+n-1)}{1.2\ldots n} \equiv \frac{(n+1)(n+2)(n+3)}{1.2.3};$$

but in estimating the number of constants with reference to the number of conditions which the locus can be made to satisfy, we must diminish this number by one, since the generality of the equation is unaltered if we divide by any one of the constants.

The number of disposable constants, so obtained, is

$$\frac{(n+1)(n+2)(n+3)}{1.2.3} - 1 \equiv n\frac{(n^2+6n+11)}{6} \equiv \phi(n).$$

$$\text{Thus}\quad \phi(2) = 9,\quad \phi(3) = 19,\quad \phi(4) = 34,$$
$$\phi(5) = 55,\quad \phi(6) = 83,\quad \text{and so on.}$$

Since, when a point is given, we may substitute its coordinates in the general equation of a given degree, and thus obtain a linear equation of condition between the constants, a surface of the third degree may be made to pass through 19 arbitrarily chosen points, and one of the fourth through 34, &c., and $\phi(n)$ arbitrarily chosen points will completely determine the position and dimensions of a surface of the n^{th} degree.

A surface of the n^{th} degree is also determined by $\phi(n)$ independent linear equations of any kind between its coefficients.

398. *All surfaces of the* n^{th} *degree which pass through* $\phi(n)-1$ *given points have a common curve of intersection.*

If $u=0$, $v=0$ be the equation of two surfaces passing through the given points, $\lambda u + \mu v = 0$ will be the equation of another

surface of the n^{th} degree which passes through the $\phi(n)-1$ given points; and since, by giving proper values to the ratio $\lambda : \mu$, this surface may be made to pass through any additional point which is not common to the two surfaces $u=0$, $v=0$, this equation will be the general equation of all surfaces which contain the $\phi(n)-1$ given points. But this equation is also satisfied by the coordinates of all points which lie on the curve of intersection of $u=0$ and $v=0$; this curve, which is of the degree n^2, is therefore the common curve of intersection of all surfaces containing the $\phi(n)-1$ points.

399. By reasoning similar to the above, it can be seen that, if a surface be of such a nature that m points or m linear equations of condition completely determine it, we may assert, that if $m-1$ such conditions be given, all surfaces of this kind which satisfy these conditions will have a common curve of intersection.

400. We shall give the name of *cluster* to the series of surfaces of the n^{th} degree determined by the equation $\lambda u + \mu v = 0$, and call the curve of the degree n^2, which is the common intersection of the surfaces, the *base* of the cluster.

We have adopted *cluster* as the equivalent of the term *faisceau* used by French writers.

401. We may remark that, if $\phi(n)-1$ points be given, it will be possible to eliminate from the general equation of the surface of the n^{th} degree all the constants but one, which will enter into the resulting equation in the first power only. This equation will then be of the form $u+\lambda v=0$, where u, v are of the n^{th} degree, and λ an undetermined constant. All surfaces represented by this equation will pass through the curve given by the equations

$$u=0, \quad v=0,$$

which curve is therefore completely determined. For example, eight points determine a curve which is the complete intersection of two conicoids.

In the case of complete intersections of surfaces the nature of

the curve is not given when the degree is given, except in the case of prime numbers, when it must be a plane curve.

For example, a curve of the twelfth degree might be the complete intersection of pairs of surfaces of the degrees (1, 12), (2, 6), (3, 4), and these different species, belonging to the same degree, would require a different number of given points to determine completely the surfaces.

The following proposition serves to obtain the number of given points sufficient to determine a surface of the n^{th} degree which, by its complete intersection with a surface of a lower degree, gives a curve of the nq^{th} degree: this is given by Plücker, but may also be proved directly by a theorem given by Cayley.*

402. *All surfaces of the n^{th} degree which pass through*

$$\phi(n) - \phi(n-q) - 1$$

given points of a surface of the q^{th} degree cut this last surface in one and the same curve.

Of $\phi(n) - 1$ given points, $\phi(p)$ lie on a surface of the p^{th} degree, whose equation is $u_p = 0$, and if the rest, viz.

$$\phi(n) - \phi(p) - 1,$$

lie on a surface of the q^{th} degree, where $n = p + q$, whose equation is $u_q = 0$, then $u_p u_q = 0$ will be one of the surfaces which contain the $\phi(n) - 1$ points, and may be obtained by giving a certain value to the ratio $\lambda : \mu$ in the equation $\lambda u + \mu v = 0$, so that

$$\lambda u + \mu v \equiv u_p u_q.$$

The curve of intersection of all the surfaces of the n^{th} degree containing these points lies on the surfaces $u_p = 0$ and $u_q = 0$.

Hence if $\phi(n) - \phi(n-q) - 1$ points be taken on any fixed surface $u_q = 0$, all surfaces of the n^{th} degree, which pass through these points, will intersect the surface of the q^{th} degree in the same curve.

Thus, if $q = 1$, the proposition is reduced to the following: All surfaces of the n^{th} degree which pass through $\frac{1}{2}n(n+3)$

* *Nouvelles Annales*, XII., p. 396.

given points in a plane determine a curve of the n^{th} degree in that plane.

If $q = 2$, the proposition becomes:

All surfaces of the n^{th} degree which pass through $n(n+2)$ points on a conicoid intersect the conicoid in the same curve.

403. When it is said that a curve or surface is determined by a certain number of points, these points must be supposed arbitrarily taken, for it is possible so to select the points that this number would not be sufficient. Thus, a plane cubic is generally determined by 9 points, but, if those be the nine points of intersection of two of such curves, an infinite number may be drawn through them. A curve of the fourth degree of one species can be determined completely by 8 arbitrary points, but if these given points be the intersections of three conicoids which have not a common curve of intersection, taking these surfaces two and two, we may obtain three curves of that species passing through the same eight points.

404. If two surfaces of the n^{th} degree pass through a curve of the nr^{th} degree situated on a surface of the r^{th} degree, they will also intersect in a curve of the $n(n-r)^{\text{th}}$ degree, situated on a surface of the $(n-r)^{\text{th}}$ degree, because one of the surfaces which passes through the intersection of the two n^{tic} surfaces will be the complex surface formed of two of the degrees r and $n-r$ respectively.

Thus, if a conicoid intersect a cubic surface in three conics, the planes of each of these will intersect the cubic in a straight line, making the complete intersection; and since the three planes form a cubic surface, part of whose curve of intersection with the original cubic surface lies on a conicoid, the three straight lines will lie in one plane.

405. The theory of *partial* intersections of surfaces was first discussed by Salmon.* Without an examination of such partial intersections it is not possible to analyze different species of curves of the same degree. If we considered only *complete*

* *Quarterly Journal*, vol. v.

intersections of surfaces, curves of the third degree could only be considered as plane curves, whereas it will be seen that they may also be *partial* intersections of conicoids.

406. *To find the surfaces of which a given curve is the partial intersection.*

In order to find the surfaces which may contain a curve of the m^{th} degree, it is observed that through $\phi(k)$ points a surface of the k^{th} degree can be made to pass. Now, the total number of points which are common to a proper curve of the m^{th} degree and such a surface, supposing the curve not to lie entirely on the surface, is mk, since this is the number of points in which k planes intersect the curve; and the law of continuity makes the statement general.

If $\phi(k) = mk + 1$, one such surface can be drawn containing the curve; if $\phi(k) > mk + 1$, two surfaces of the k^{th} degree can be drawn, and therefore an infinite number. Thus, for a curve of the third degree, if $k = 2$, $\phi(k) = 9 > 3.2 + 1$, hence an infinite number of conicoids may be drawn containing any curve of the third degree.

When $\phi(k) = mk + 1$, one surface of the k^{th} degree contains the curve, and another of the $\overline{k+1}|^{th}$ must also contain it, for $\phi(k+1) - \phi(k)$, $= \frac{1}{2}(k+2)(k+3)$, therefore

$$\phi(k+1) - \{m(k+1) + 1\} = \tfrac{1}{2}(k+2)(k+3) - m,$$

which is always positive.

Modifications are required when the surfaces are not proper surfaces. Salmon gives as examples of this modification a plane curve of the third degree through which it is possible to describe an infinite number of conicoids, but since each conicoid must necessarily consist of the plane of the curve and an arbitrary plane, the intersection of the plane and conicoid will not determine the curve. Again, if a curve of the fifth degree, which, according to the above laws, ought necessarily to be determined by surfaces of the third degree, lie entirely on a conicoid, every surface of the third degree which contains the curve may be a compound of the conicoid and a plane, and we must advance to surfaces of the fourth degree to determine the curve.

If a curve be given of the m^{th} degree, and k, l be the lowest degrees of surfaces upon which it can lie, any surface of the k^{th} degree constructed to pass through $mk+1$ points will contain the curve, and similarly for the other surface.

If $ml+1$ points known to lie on the curve be given, and $l > k$, all the rest can be found.

407. The number of *arbitrary* points through which a curve of the m^{th} degree can be drawn cannot exceed a certain superior limit which is easily determined, for suppose k arbitrary points to be given, and a cone to be constructed containing the curve, and having its vertex in one of the assumed points, the degree of this cone will be $m-1$, since any plane through the vertex must contain $m-1$ points of the curve besides the vertex, and therefore $m-1$ generating lines of the cone, and the number of its generating lines sufficient for its complete determination is the same as that of the number of points necessary to determine a plane curve of the $\overline{m-1}\rceil^{\text{th}}$ degree, viz. $\dfrac{m(m+1)}{2}-1$.

The greatest value of k for which such a cone can be constructed is $\dfrac{m(m+1)}{2}$; this is therefore a superior limit, although lower limits to the number k may be obtained in general from other considerations.

Thus, a curve of the third degree cannot be made to pass through more than six arbitrarily chosen points.

408. *If $\phi(n)-2$ points be given, all surfaces of the n^{th} degree, which can be drawn through these points, will pass through $n^3-\phi(n)+2$ additional fixed points.*

Let $u=0$, $v=0$, $w=0$ be the equations of three surfaces of the n^{th} degree which pass through $\phi(n)-2$ points, and which have not a common curve of intersection, they will pass through n^3 common points, and $\lambda u+\mu v+\nu w=0$ is the equation of another surface of the n^{th} degree, which passes through the same points, and by giving different values to $\lambda : \mu : \nu$ we can obtain *all* surfaces which pass through these points. Any surface will be particularized when two points are given which do not lie on all

three of the surfaces, or both on the same two; and all such surfaces will contain $n^3 - \phi(n) + 2$ common points besides the given points.

Thus, all conicoids which pass through seven points will pass through a fixed eighth, as is easily seen when each conicoid consists of two parallel planes, the seven points being angular points of a parallelepiped.

A surface of the third degree, drawn through 17 points, passes through 10 others.

409. The following propositions, connected with this part of the subject, are of importance in some investigations in which it is required to determine the number of points of intersection of three surfaces: the surfaces under consideration in particular cases may have common lines in any degree of multiplicity, and it becomes necessary to determine to how many points of intersection these lines are equivalent.

410. *Three surfaces of the m^{th}, n^{th}, and p^{th} degrees, contain a multiple straight line in the degrees of multiplicity μ, ν, and ϖ respectively; to find the number of points of intersection to which this multiple line corresponds.*

The number of points of intersection of three surfaces will be unaltered, if we suppose each surface to degenerate into a set of proper surfaces of inferior degrees, so long as the sum of the degrees of the set is the degree of the surface so broken up.

We will, therefore, suppose the surface of the m^{th} degree to consist of μ planes, and of a proper surface of the $(m-\mu)^{\text{th}}$ degree; and similarly for the others.

The whole number mnp of points of intersection will then be made up of intersections (1) of the three proper surfaces, (2) of proper surfaces from two systems with planes from the remaining system, (3) of a proper surface of one system with planes from the two remaining systems, and (4) of planes from the three systems: the numbers of these intersections are

(1) $(m-\mu)(n-\nu)(p-\varpi)$, (2) $(n-\nu)(p-\varpi)\mu + \ldots$,

(3) $(m-\mu)\nu\varpi + \ldots$, (4) $\mu\nu\varpi$.

If now we suppose all the planes to pass through the same straight line, we shall have the case of surfaces with multiple lines; and those of the mnp points, which will lie on the multiple line, will be clearly taken from the groups (3) and (4).

The multiple line therefore corresponds to the number of points

$$(m-\mu)\,\nu\varpi + (n-\nu)\,\varpi\mu + (p-\varpi)\,\mu\nu + \mu\nu\varpi$$
$$\equiv m\nu\varpi + n\varpi\mu + p\mu\nu - 2\mu\nu\varpi,$$

which coincides with the particular case given by Salmon.*

411. *Three surfaces of the m^{th}, n^{th}, and p^{th} degrees have a common curve line of the μ^{th}, ν^{th}, ϖ^{th} degrees of multiplicity respectively, the curve being the intersection of two surfaces of the degrees k and l: to find the number of points to which this multiple line corresponds.*

Let the surfaces be broken up into proper surfaces, and the multiple lines be thrown out of gear.

The first shall be composed of μ surfaces of the degree k and one of the degree $m-\mu k$, the second of ν surfaces of the degree k, ν of degree l and one of the degree $n-\nu k-\nu l$, the third of ϖ of the degree l, and one of the degree $p-\varpi l$.

The number of points which lie on the intersection of surfaces of the degrees k and l will be

$$(m-\mu k)\,\nu k.\varpi l + (n-\nu l-\nu k)\,\mu k.\varpi l + (p-\varpi l)\,\mu k.\nu l$$
$$+\,\varpi l.\mu k.(\nu k+\nu l) \equiv lk\,\{m\nu\varpi + n\varpi\mu + p\mu\nu - \mu\nu\varpi\,(l+k)\},$$

which is the number of points required, coinciding with the result of the preceding proposition when $l=k=1$.

Application to the Four-point System.

412. It is easy to express in the language of four-point coordinates the results of this chapter.

Thus, a surface of the n^{th} class is determined if $\phi(n)$ tangent planes be given.

If surfaces of the n^{th} class be drawn touching $\phi(n)-1$ tan-

* *Camb. and Dub. Math. Jour.*, II., p. 71.

gent planes, they will be touched by one common developable surface.

If three surfaces of the n^{th} class touch $\phi(n) - 2$ given planes, they will touch $n^3 - \phi(n) + 2$ additional fixed planes.

Similarly for other theorems.

In illustration of the points which have been considered in this chapter, relating to the intersection of surfaces, we give here some elementary properties of cubic and quartic curves.

Cubic Curves.

413. If two conicoids have a common generating line, any plane which does not contain this generating line will intersect the two conicoids in two conics which have four points in common, one of which will be in the generating line; hence the curve which with the generating line forms the complete intersection of the conicoids, being met by an arbitrary plane in three points, is a curve of the third degree; such a curve is called a *cubic curve.*

Conversely, if we take any seven points upon a given cubic curve and an eighth on any chord of the curve, we can make an infinite number of conicoids pass through these eight points, which will have for their common curve of intersection the cubic curve and the chord, for each conicoid meets the curve in seven points and the chord in three, and therefore contains both entirely.

414. *A cubic curve, which is the intersection of two conicoids having a common generating line, intersects all the generating lines of the same system as the common line in two points, and those of the opposite system in one point only.*

Call the two conicoids A and B, and the common line L. Any generating line of A intersects B in two points, neither of which will lie on L, if it be of the same system as L, but one will lie on L, if it be of the system opposite to that of L; but the points which do not lie on L must lie on the cubic curve, which proves the proposition.

415. *The common generating line of two conicoids which determine a cubic curve is twice crossed by the curve.*

A plane which contains the common generating line intersects each of the conicoids in a generating line of the opposite system, and these two lines intersect in one point only; but the plane contains three points of the curve; hence two of the three points must lie on the common generating line.

416. *When two cubic curves lie on a given conicoid, to find the number of points in which they intersect.*

Each of the cubic curves is the partial intersection of the given conicoid with another which has a common generating line with it.

Call the three conicoids A, B, and B', and the curves C, C', and let the complete intersection of A and B be the curve C and the line L, and let that of A and B' be the curve C' and the line L'.

The eight points which are common to A, B, and B' must be the intersections of the complex curves CL and $C'L'$; and two distinct cases arise according as L, L' are of the same or of opposite systems.

If they be of the same system, L will meet B' in two points both of which will be on C'; L' and C will intersect in two points, therefore C and C' will intersect in four points.

If they be of opposite systems the two points in which L intersects B' will lie one on L' and the other on C'; hence L, L'; L, C'; and L', C will intersect in three points, and therefore C and C' in the five remaining points.

Quartic Curves.

417. The intersection of two conicoids is a quartic curve; since a plane must meet the two conicoids in two conics which intersect in four points; but this is a particular kind of quartic curve. An arbitrary quartic curve will intersect an arbitrary conicoid in eight points, and only one conicoid can be constructed which will contain nine points of the curve, and therefore the entire curve.

The general quartic curve may therefore be considered as the partial intersection of a conicoid and a cubic surface drawn through thirteen points of the curve, and the remaining portion of the complete intersection must be either (i) two straight lines which do not intersect, or (ii) a conic which may be two intersecting straight lines.

i. In the first case a generating line of the conicoid which is of the same system as the two straight lines common to the two surfaces, meets the cubic surface in three points which must be on the quartic curve, while one of the opposite system meets the cubic surface in one point only, besides the points in which it cuts the two common lines, and therefore intersects the quartic curve once.

ii. In the second case every generator of the conicoid meets the common conic in one point, therefore two of the three points in which it intersects the cubic surface lie on the quartic curve.

If $u_2 = 0$ be the equation of a conicoid containing the quartic curve, $u_1 = 0$ that of the plane of the common conic, the equation of the cubic surface must be of the form $v_1 u_2 + u_1 v_2 = 0$, and the quartic curve, in this case, must be of the particular kind which is the base of a cluster of conicoids, viz. that determined by the equation $\lambda u_2 + \mu v_2 = 0$ for all arbitrary values of λ and μ.

418. *To find the number of points of intersection of two quartic curves which both lie on the same cubic surface.*

Let the surface be denoted by S_3 and the conicoids which contain the two curves by S_2 and S_2', and suppose the remaining parts of the complete intersections to be two non-intersecting lines, so that the complete intersection of S_2 and S_3 is C_4, L, M, and that of S_2' and S_3 is C_4', L', M'.

The three surfaces S_2, S_2', and S_3 intersect in 12 points, and since L intersects S_2' in 2 points, and similarly for the other lines, 8 of the 12 points lie on the extraneous lines, and the two curves C_4, C_4' intersect in four points.

This supposes that the lines L, L' do not intersect, but if they intersect, the modification is easily made; for example, if the four lines form a skew quadrilateral, the number of points

belonging to these lines will be reduced to four, and C_4, C_4' will intersect in eight points.

By similar reasoning, if the remainder of the intersection of S_2 and S_3, or of S_2' and S_3 be a conic, it can be shewn that C_4 and C_4' will generally intersect in four points; and if the three surfaces all contain the same conic, by Art. 411 this will count as eight points of intersection, and therefore C_4, C_4' will intersect in the four remaining points.

XVII.

(1) Every cone containing a curve of the third degree, in which the vertex lies, is of the second degree.

(2) Prove that an infinite number of curves of the third degree can be drawn through five points arbitrarily chosen in space, but that six determine the curve: what limitations are necessary that such a curve shall pass through the points?

(3) Through a curve of the third degree, and a straight line meeting the curve in one point only, a conicoid can be drawn, of which the generating lines, which do not intersect the given line, meet the curve each in two points.

(4) Through any point in space a straight line can be drawn which meets a curve of the third degree, not a plane curve, in two points.

(5) If P, Q be two points on a cubic curve, all the conicoids which contain the curve and the chord PQ have common tangent planes at P and Q.

(6) A conicoid can be drawn through a given chord of a cubic curve containing the curve and touching a given plane through the chord at a given point of the chord.

(7) The projection upon any plane of a curve of the third degree, not plane, by straight lines drawn from a given point, is a curve of the third degree having a double point.

(8) No straight line can cut a curve of the n^{th} degree, not plane, in more than $n - 1$ points.

(9) The locus of the centres of a cluster of conicoids is a cubic curve.

(10) Three conicoids which have a common generating line meet only in four points besides the generating line.

(11) Through five points of a conicoid, we can draw two curves of the third degree lying entirely in the conicoid.

(12) A quartic curve is the intersection of two conicoids, prove that a cubic surface can be constructed which contains the curve and two given conics, one on each conicoid, if these conics do not lie in the same plane.

(13) Shew that, if normals be drawn to a conicoid from every point of a straight line, their feet will lie on a quartic curve.

(14) Among the conicoids forming a cluster there are four cones, real or imaginary; each of these cones has four of its sides tangents to the curve which is the base of the cluster, and the four points of contact are in one plane.

(15) Through a chord AB of a quartic curve, which is the base of a cluster of conicoids, a plane is drawn determining a second chord ab; shew that, as the plane turns round AB, the chord ab generates a conicoid; shew also that a plane which passes through ab and a fixed point E of the curve also passes through a fixed chord EF.

(16) The projection of the base of a cluster of conicoids on a plane is a curve of the fourth degree having two double points, real or imaginary.

(17) Two quartic curves which lie on the same hyperboloid, each intersect one system of generating lines in three points, prove that the curves intersect in six points if the generating lines be of the same system for both curves, and in ten points if the systems be opposite.

(18) Through a given straight line planes are drawn touching the sections of a conicoid made by a plane passing through a second straight line; shew that the locus of the points of contact is a quartic curve passing through the four points in which the two straight lines intersect the conicoid, and four of whose tangents intersect both of these straight lines.

(19) If three straight lines be the complete intersection of a cubic surface with a plane, three planes through these lines will intersect the surface in three conics; prove that one conicoid can be drawn containing the three conics.

(20) Find the number of points in which two curves of the fifth degree on the same surface of the third degree, and on two conicoids, intersect.

(21) The eight points given by the equations $lx^2 = my^2 = nz^2 = rw^2$, are so related that any conicoid passing through seven of them will pass through the eighth.

(22) Three cones of the same degree have their vertices on a straight line, and two of their three curves of intersection are plane curves; shew that the third curve is also plane, and that the planes of the three curves intersect in a straight line.

CHAPTER XVIII.

TANGENT LINES AND PLANES. NORMALS. SINGULAR POINTS. SINGULAR TANGENT PLANES. POLAR EQUATION OF TANGENT PLANE. ASYMPTOTES.

419. IN this chapter we shall, for reasons given in the preface, confine ourselves to the consideration of surfaces whose equations are given in Cartesian coordinates, and in discussing singularities of contact we shall only consider those of a simpler kind, reserving for a later portion of the work those which are interesting merely as subjects of pure geometry.

420. It will be convenient to state here, that we shall often employ the following notation: when the function $F(x, y, z)$, for which we shall write F, is used, U, V, W will be written for $\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{dz}$, and u, v, w, u', v', w' for $\frac{d^2F}{dx^2}, \frac{d^2F}{dy^2}, \frac{d^2F}{dz^2}, \frac{d^2F}{dydz}, \frac{d^2F}{dzdx}, \frac{d^2F}{dxdy}$, and when $z = f(x, y)$, p, q, r, s, t will be written for $\frac{dz}{dx}, \frac{dz}{dy}, \frac{d^2z}{dx^2}, \frac{d^2z}{dxdy}, \frac{d^2z}{dy^2}$.

421. *To find the relation between the direction-cosines of a tangent to a surface at a given ordinary point of a surface.*

Let the equation of the surface be $F \equiv F(\xi, \eta, \zeta) = 0$, ξ, η, ζ being the current coordinates of a point, and let (x, y, z) be the point P at which the line is a tangent.

The equations of a line through P, whose direction-cosines are λ, μ, ν, are

$$\frac{\xi - x}{\lambda} = \frac{\eta - y}{\mu} = \frac{\zeta - z}{\nu} = r. \tag{1}$$

At the points where this line meets the surface, the values of r

are given by the equation $F(x+\lambda r,\ y+\mu r,\ z+\nu r)=0$, and, since $F(x, y, z)=0$, this equation may be written

$$r\left(\lambda\frac{d}{dx}+\mu\frac{d}{dy}+\nu\frac{d}{dz}\right)F+\frac{r^2}{2}\left(\lambda\frac{d}{dx}+\mu\frac{d}{dy}+\nu\frac{d}{dz}\right)^2F+\dots$$

$$+\frac{r^s}{\lfloor s}\left(\lambda\frac{d}{dx}+\mu\frac{d}{dy}+\nu\frac{d}{dz}\right)^sF+\dots=0. \qquad (2)$$

One value of r is zero, whatever be the direction of the line (1), since P is on the surface, but if we so choose the direction of the line that

$$\lambda\frac{dF}{dx}+\mu\frac{dF}{dy}+\nu\frac{dF}{dz}=0, \qquad (3)$$

a second value of r will vanish; therefore for this direction another point Q will become coincident with P, and the line will be a tangent line.

At an ordinary point $\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{dz}$ do not all vanish, but there may exist points on a surface for which this does happen; such points are called singular points: we shall presently consider this peculiarity.

422. *To find the equations of the tangent plane and the normal to a surface at an ordinary point.*

The equation of the locus of all the tangent lines which can be drawn through an ordinary point is found by eliminating λ, μ, ν between the equations (1) and (3), which gives

$$(\xi-x)\frac{dF}{dx}+(\eta-y)\frac{dF}{dy}+(\zeta-z)\frac{dF}{dz}=0,$$

shewing that the tangent lines all lie in a plane, which is called the *tangent plane*.

The normal is perpendicular to this plane, and its equations are

$$\frac{\xi-x}{\frac{dF}{dx}}=\frac{\eta-y}{\frac{dF}{dy}}=\frac{\zeta-z}{\frac{dF}{dz}}=\frac{r}{\sqrt{\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2\right\}}};$$

the equation (3) represents that the normal is perpendicular to every tangent line.

423. *To find the number of normals which can be drawn from a given point to a surface of the n^{th} degree.*

Let $F(\xi, \eta, \zeta) = 0$ be the surface. The number of normals will be the same from whatever point they be drawn, the number may therefore be found by investigating the number of normals which can be drawn from a point at an infinite distance, which we may assume in Ox produced.

The number will therefore be equal to the number of normals parallel to Ox, together with the number of normals to the section by a plane at an infinite distance.

If (x, y, z) be the foot of a normal parallel to Ox, $F'(y) = 0$, $F'(z) = 0$, which combined with the equation $F(x, y, z) = 0$ give $n(n-1)^2$ solutions.

Again, any plane section of the surface will be of the n^{th} degree, and the number of normals drawn to any curve $f(x, y) = 0$ of the n^{th} degree is, in like manner, the number of normals parallel to Ox, together with the normals which can be drawn to points at an infinite distance, the number of the latter is n, and the number of normals parallel to Ox is given by the number of solutions of $f'(y) = 0$, and $f(x, y) = 0$, which is $n(n-1)$; hence, the number of normals to the plane section at an infinite distance is n^2.

Therefore, the number of normals which can be drawn to the surface from any point

$$= n(n-1)^2 + n^2 = n^3 - n^2 + n.$$

424. *To obtain the form of the equation of the tangent plane when $F(\xi, \eta, \zeta)$ is represented as the sum of a series of homogeneous functions.*

$$\text{Let } F(x, y, z) = F_n + F_{n-1} + \ldots + F_1 + c$$

where F_s denotes a homogeneous function of the s^{th} degree in x, y, z; then, by a known property of homogeneous functions,

$$\left(x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz}\right) F_s = sF_s;$$

therefore the equation of the tangent plane may be written

$$\xi\frac{dF}{dx} + \eta\frac{dF}{dy} + \zeta\frac{dF}{dz} = nF_n + (n-1)F_{n-1} + \ldots + F_1,$$

or, since $F_n + F_{n-1} + \ldots + c = 0$,

$$\xi\frac{dF}{dx} + \eta\frac{dF}{dy} + \zeta\frac{dF}{dz} + F_{n-1} + 2F_{n-2} + \ldots + (n-1)F_1 + nc = 0.$$

425. *To find the equation of a tangent plane to a surface, when the direction of the plane is given.*

Let (l, m, n) be the given direction, and $l\xi + m\eta + n\zeta = p$ the equation of a tangent plane to the surface $F(\xi, \eta, \zeta) = 0$; then if (x, y, z) be the point of contact, since this equation must be identical with

$$\xi\frac{dF}{dx} + \eta\frac{dF}{dy} + \zeta\frac{dF}{dz} = x\frac{dF}{dx} + y\frac{dF}{dy} + z\frac{dF}{dz},$$

we have

$$\frac{1}{l}\frac{dF}{dx} = \frac{1}{m}\frac{dF}{dy} = \frac{1}{n}\frac{dF}{dz} = \frac{1}{p}\left(x\frac{dF}{dx} + y\frac{dF}{dy} + z\frac{dF}{dz}\right),$$

and these equations, with that of the surface, give the coordinates of the points of contact of any tangent plane in the given direction, and also determine a relation between l, m, n and p, such as was found in Art. 253 in the case of a conicoid; this relation is the tangential equation with the Boothian coordinates $\frac{l}{p}, \frac{m}{p}, \frac{n}{p}$.

426. *To find the locus of the points of contact of tangent planes drawn to a given surface from a given point.*

Let $F \equiv F(\xi, \eta, \zeta) = 0$ be the equation of the given surface of the n^{th} degree, and let (f, g, h) be the given point. If (x, y, z) be one of the points of contact, the tangent plane to the surface at (x, y, z) must pass through (f, g, h).

This gives the condition

$$(f-x)\frac{dF}{dx} + (g-y)\frac{dF}{dy} + (h-z)\frac{dF}{dz} = 0, \qquad (1)$$

which, combined with the equation of the surface, determines the required locus, or the *curve of contact*.

It has been shewn, (Art. 421), that $x\frac{dF}{dx} + y\frac{dF}{dy} + z\frac{dF}{dz}$ may by means of the equation of the surface be reduced to an ex-

pression of the $(n-1)^{\text{th}}$ degree in x, y, z; the equation (4) so reduced gives a surface of the $(n-1)^{\text{th}}$ degree, called the *first polar*, whose intersection with the given surface is the curve of contact.

The curve of contact for any conicoid is therefore a conic, the first polar being in this case a plane.

The general theory of polars will be considered hereafter.

Singular Points.

427. *To find the relation between the direction-cosines of a tangent line at a singular point.*

Since at a singular point P, $\dfrac{dF}{dx}$, $\dfrac{dF}{dy}$ and $\dfrac{dF}{dz}$ separately vanish, the coefficient of r in equation (2) vanishes for all values of λ, μ, ν, which shews that the line (1) meets the surface in two coincident points, in whatever direction it be drawn through P; in this case we find the direction of any tangent line by taking a point Q near the double point P, and moving it up to P until a third value of r vanishes, the direction of PQ will then be that of a tangent line, and the relation between its direction-cosines will be

$$\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy} + \nu \frac{d}{dz}\right)^2 F = 0,$$

or, written in full,

$$u\lambda^2 + v\mu^2 + w\nu^2 + 2u'\mu\nu + 2v'\nu\lambda + 2w'\lambda\mu = 0. \qquad (5)$$

If all the partial differential coefficients as far as those of the $(s-1)^{\text{th}}$ order vanish, the equation (3) will have s roots equal to zero, and the point will be a multiple point of the s^{th} degree; it is easily seen that the direction-cosines of any tangent line will satisfy the equation

$$\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy} + \nu \frac{d}{dz}\right)^s F = 0;$$

which shews that the tangent lines all lie in a cone whose vertex is the point P. This cone is called a *tangent cone*.

428. *To find the equation of the tangent cone at a multiple point.*

If the multiple point be of the s^{th} degree, the direction-cosines will satisfy the equation

$$\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy} + \nu \frac{d}{dz}\right)^s F = 0,$$

and the equation of the tangent cone is found by eliminating λ, μ, ν between this equation and the equations (1), we thus obtain

$$\left\{(\xi - x)\frac{d}{dx} + (\eta - y)\frac{d}{dy} + (\zeta - z)\frac{d}{dz}\right\}^s F = 0,$$

where it must be remembered that in the performance of the operation indicated $\xi - x$, $\eta - y$ and $\zeta - z$ must be treated as constant, in other words, the symbol of operation must be expanded before the differentiations are performed.

429. *To find the equation of the normal cone at a double point.*

The equation of the tangent cone at a double point is, by (5),

$$u(\xi - x)^2 + v(\eta - y)^2 + w(\zeta - z)^2$$
$$+ 2u'(\eta - y)(\zeta - z) + 2v'(\zeta - z)(\xi - x) + 2w'(\xi - x)(\eta - y) = 0,$$

and that of the tangent plane at any point of a generating line of this cone whose coordinates are $x + \lambda r$, $y + \mu r$, $z + \nu r$ is

$$(u\lambda + w'\mu + v'\nu)(\xi - x) + (w'\lambda + v\mu + u'\nu)(\eta - y)$$
$$+ (v'\lambda + u'\mu + w\nu)(\zeta - z) = 0;$$

hence, if $\dfrac{x - \xi}{\lambda'} = \dfrac{y - \eta}{\mu'} = \dfrac{z - \zeta}{\nu'}$ be the equation of the normal to this plane at (x, y, z),

$$\frac{u\lambda + w'\mu + v'\nu}{\lambda'} = \frac{w'\lambda + v\mu + u'\nu}{\mu'} = \frac{v'\lambda + u'\mu + w\nu}{\nu'} = \rho;$$

$$\therefore \quad u\lambda + w'\mu + v'\nu - \lambda'\rho = 0,$$
$$w'\lambda + v\mu + u'\nu - \mu'\rho = 0,$$
$$v'\lambda + u'\mu + w\nu - \nu'\rho = 0,$$
$$\lambda'\lambda + \mu'\mu + \nu'\nu \qquad = 0;$$

$$\therefore \quad \begin{vmatrix} u, & w', & v', & \lambda' \\ w', & v, & u', & \mu' \\ v', & u', & w, & \nu' \\ \lambda', & \mu', & \nu', & 0 \end{vmatrix} = 0, \qquad (5)$$

and the equation of the locus of the normals to all the tangent planes to the tangent cone is

$$p(\xi - x)^2 + q(\eta - y)^2 + r(\zeta - z)^2$$
$$+ 2p'(\eta - y)(\zeta - z) + 2q'(\zeta - z)(\xi - x) + 2r'(\xi - x)(\eta - y) = 0,$$

where $p = vw - u'^2$, $p' = v'w' - uu'$, &c.

430. The condition that the tangent cone shall degenerate into two tangent planes is

$$T = \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} = 0, \text{ Art. 88,}$$

and in this case, the equation (5) becomes

$$\frac{1}{p}(p\lambda' + r'\mu' + q'\nu')^2 = 0, \text{ Art. 364}$$

so that the normal cone degenerates into two coincident planes; this may be accounted for geometrically in the following manner: the generating lines of the normal cone are each perpendicular to the plane containing two of the generating lines of the tangent cone taken indefinitely near to one another; if then the tangent cone become two planes, we can take the two generating lines on one plane, which gives a normal to that plane; or we may take one on each close to the line of intersection of the planes, which will give a normal in any direction we please in the plane perpendicular to the line of intersection, and a double plane will be formed, because these two generators may be on either side of the double point.

The equations of the line of intersection of the two tangent planes will, by Art. 89, be

$$(v'w' - uu')(\xi - x) = (w'u' - vv')(\eta - y) = (u'v' - ww')(\zeta - z),$$
$$\text{or } p'(\xi - x) = q'(\eta - y) = r'(\zeta - z),$$

and that of one of the coincident planes in which the normals lie

$$p(\xi - x) + r'(\eta - y) + q'(\zeta - z) = 0,$$

and this plane is perpendicular to the line of intersection, since $pp' = q'r'$ (Art. 363).

If $T = 0$, $N = 0$ be the respective conditions that the tangent and normal cones may become two planes, and P, Q, R, P', Q', R' be the minors of N, $N = Pp + R'r' + Q'q'$, but

$$P = qr - p'^2 = uT, \quad \text{Art. 363},$$

$$P' = q'r' - pp' = u'T, \text{ \&c.};$$

$$\therefore \; N = T(up + w'r' + v'q') = T^2.$$

431. *To find the equation of the tangent plane and normal at any point of the surface given by the equation* $\zeta = f(\xi, \eta)$.

Let a line be drawn through x, y, z, whose equations are

$$\xi - x = m(\zeta - z), \quad \eta - y = n(\zeta - z),$$

the points in which this line meets the surface are those for which ζ is given by the equation

$$\zeta - z = f\{x + m(\zeta - z), \; y + n(\zeta - z)\} - f(x, y)$$

$$= (mp + nq)(\zeta - z) + \tfrac{1}{2}(rm^2 + 2smn + tn^2)(\zeta - z)^2 + \ldots,$$

and if the line be a tangent line two values of $\zeta - z$ are zero; therefore $1 = mp + nq$, and eliminating m and n by means of the equations of the line, we obtain the locus of the tangent lines $\zeta - z = p(\xi - x) + q(\eta - y)$, which is the equation of the tangent plane, unless p and q assume the indeterminate form $\frac{0}{0}$. The equation is deducible immediately from that of Art. 422 by means of the equations

$$\frac{dF}{dx} + p\frac{dF}{dz} = 0 \text{ and } \frac{dF}{dy} + q\frac{dF}{dz} = 0.$$

The equations of the normal are

$$\xi - x + p(\zeta - z) = 0 \text{ and } \eta - y + q(\zeta - z) = 0.$$

432. Before we consider the properties of the curve of intersection of a surface with its tangent plane, we should notice that among all the tangent lines drawn at an ordinary point, whose locus is the tangent plane, there are two whose direction-

cosines satisfy the equation $\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy} + \nu \frac{d}{dz}\right)^2 F = 0$ as well as $\lambda \frac{dF}{dx} + \mu \frac{dF}{dy} + \nu \frac{dF}{dz} = 0$, and that for these lines three points become coincident, so that they have a closer contact with the surface than any of the other tangent lines; these tangent lines are called inflectional tangents. In the case of a conicoid three points on one line cannot coincide, unless the line lie entirely in the surface, and the two particular tangent lines are the two generating lines which pass through the point of contact.

Among the tangent lines at a double point it will be seen similarly that there are generally six which have a closer contact than the rest.

433. *Geometrical explanation of the nature of the intersection of a surface with its tangent plane at any point.*

Every plane intersects a surface of the n^{th} degree in a curve which is of the same degree; hence a tangent plane at any point intersects the surface in a curve of the n^{th} degree, passing through the point of contact.

Now when a tangent plane exists, since it is the locus of the tangent lines at the point of contact, and each of these tangent lines contains two points which coincide in the point of contact, it follows that any line, drawn in the tangent plane through the point of contact, meets the curve of intersection in two points at the point of contact.

The point of contact is, therefore, a singular point in the curve of intersection.

The singular point may be either a conjugate point, as in the case of contact with an ellipsoid; or a multiple point, as in the case of an hyperboloid of one sheet; or a point through which two coincident lines pass, as in the case of a cylinder.

If the surface be of the second degree the curve of intersection will be of the second degree, and, since it must contain a singular point, the only admissible lines of intersection will be either an indefinitely small circle or ellipse, er else two straight lines which cross one another, or are coincident.

434. *If a plane intersect a surface in a curve which contains a singular point, the plane will generally be a tangent plane to the surface at that singular point.*

For a straight line drawn in any direction in the plane through a singular point, meets the surface in two coincident points, and therefore generally satisfies the condition of being a tangent line to the surface.

If the point which is a singular point in the curve of intersection be also a singular point in the surface, the condition of passing through two coincident points will not be sufficient to define a tangent line.

Thus, if at any point of a surface there be a conical tangent, there may be a singular point in the curve of intersection of a plane through the vertex of the conical tangent, which will not make the cutting plane a tangent plane at the multiple point.

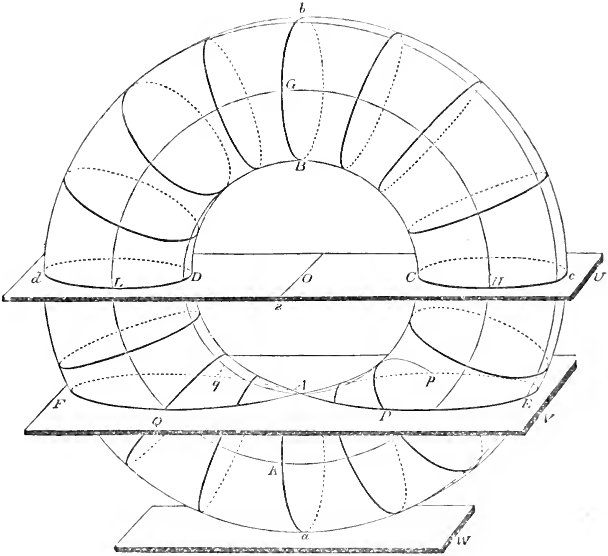
435. The form of the curve of intersection of a surface with the tangent plane at any point may be illustrated by taking the case of an anchor ring, supposed to be generated by the revolution of a circle about an axis in its plane not intersecting the circle.

The figure represents the ring, with the generating circle in different positions as it revolves about the axis Oz.

The plane U is drawn through the axis Oz, intersecting the surface in the circles CHc, DLd.

Suppose this plane to move, parallel to itself, towards the position V, the closed curves in which it intersects the surface become elongated until they meet one another in the point A, forming for the position V of the plane a figure of eight, viz. $EPAqFQAp$ which has a double point at A. Here we observe that the concavities of the circles AKa and $ACBD$, which are sections by planes perpendicular to V and to each other, lie in opposite directions with regard to the plane V, and that the tangent lines at A lie in that plane, which is therefore the tangent plane at A; and it is a tangent plane at no other point of the curve of intersection.

The sections by planes through A perpendicular to V change



the directions of their concavities as they pass from the position AKa to $ACBD$, when they pass through the tangents to the branches pAQ, PAq at the multiple point.

If the plane move past V to the position W, the curve of intersection will gradually assume an oval form, which will degenerate into a conjugate point at a.

It is clear also that a plane may meet the ring in the circle $GHKL$, in which case it is a tangent plane at every point of the curve in which it meets the surface; this curve is composed of two coincident circles, as may be seen by moving the plane inwards parallel to itself.

It will be shewn also, that a tangent plane, drawn through a line COD perpendicular to Oz, intersects the ring in two circles.

436. *To find the equations of the tangent line at any point of the curve of intersection of a surface with its tangent plane.*

Let the equation of the surface be $F(\xi, \eta, \zeta) = 0$; that of the tangent plane at (x, y, z) will be

$$(\xi - x)\frac{dF}{dx} + (\eta - y)\frac{dF}{dy} + (\zeta - z)\frac{dF}{dz} = 0,$$

$$\text{or } (\xi - x)F'(x) + (\eta - y)F'(y) + (\zeta - z)F'(z) = 0. \quad (1)$$

Let the equation, of the tangent line at any point (x', y', z') of the curve of intersection be

$$\frac{\xi - x'}{\lambda} = \frac{\eta - y'}{\mu} = \frac{\zeta - z'}{\nu},$$

since this line lies in (1),

$$\lambda F'(x) + \mu F'(y) + \nu F'(z) = 0, \quad (2)$$

and since it meets the surface in two coincident points at (x', y', z'),

$$\lambda F'(x') + \mu F'(y') + \nu F'(z') = 0; \quad (3)$$

these two equations determine $\lambda : \mu : \nu$ when (x', y', z') is an ordinary point on the curve and the surface.

437. *To find the singular points of the curve of intersection with the tangent plane at any point.*

If the point be a singular point on the curve of intersection, any line drawn through this point will have two points coincident at the point considered; hence, the two equations obtained in the preceding article will be satisfied by an infinite number of values of $\lambda : \mu : \nu$; this will happen in any of the following three cases:

(i) When $F'(x)$, $F'(y)$ and $F'(z)$ vanish simultaneously, which occurs when there is a singular point at (x, y, z), in which case there are an infinite number of tangent planes.

(ii) When $F'(x')$, $F'(y')$ and $F'(z')$ vanish simultaneously, in which case (x', y', z') is a singular point on the surface.

(iii) When $\dfrac{F'(x')}{F'(x)} = \dfrac{F'(y')}{F'(y)} = \dfrac{F'(z')}{F'(z)}$, in which case the tangent plane at (x, y, z) is a tangent plane at (x', y', z') also.

In case (i) one of the tangent planes to the tangent cone touching it along a generating line (λ', μ', ν') must be the plane considered, and the equation (2) must be replaced by

$$(u\lambda' + w'\mu' + v'\nu')\,\lambda + (w'\lambda' + v\mu' + u'\nu')\,\mu + (v'\lambda' + u'\mu' + w\nu')\,\nu = 0, \text{ (Art. 429)},$$

thus the ratio $\lambda : \mu : \nu$ will be determined, except in cases where (x', y', z') is a singular point on the surface, or where the tangent plane considered is also a tangent plane to the surface at (x', y', z').

In case (ii) a third point at least must be coincident with (x', y', z'), and the equation (3) must be replaced by

$$\left(\lambda\frac{d}{dx'} + \mu\frac{d}{dy'} + \nu\frac{d}{dz'}\right)^s F(x', y', z') = 0,$$

where s is 2, 3, ... according to the degree of multiplicity of the singular point (x', y', z').

In case (iii), if neither (x, y, z) nor (x', y', z') be singular points of the surface, the equations which determine $\lambda : \mu : \nu$ will be $\lambda F'(x) + \mu F'(y) + \nu F'(z) = 0$,

$$\text{and}\quad \left(\lambda\frac{d}{dx'} + \mu\frac{d}{dy'} + \nu\frac{d}{dz'}\right)^2 F(x', y', z') = 0,$$

whether (x', y', z') be coincident with (x, y, z) or not.

This case includes the singular tangent plane, a portion of whose curve of intersection consists of two coincident curve lines, which will be considered immediately.

Ruled Surfaces.

438. The student is already familiar with certain surfaces which are capable of being generated by straight lines, or through every point of which some straight line may be drawn which will coincide, throughout its length, with the surface.

For example,—a plane, a cone, a cylinder, an hyperboloid of one sheet, an hyperbolic paraboloid.

He is aware that any portion of two of these, the cone and the cylinder, may, if supposed perfectly flexible, be developed into a plane without tearing or rumpling.

We shall now give some account of the general character of surfaces which have this property, distinguishing them from those which, although capable of being generated by the motion of a straight line, are incapable of development into a plane.

439. Def. A *Ruled Surface* is a surface which can be generated by the motion of a straight line; or a surface through every point of which a straight line can be drawn, which will lie entirely in the surface.

A ruled surface, on which each generating line intersects that which is next consecutive, is called a *Developable Surface*, or *Torse*.

A ruled surface, on which consecutive generating lines do not intersect, is called a *Skew Surface*, or *Scroll*.

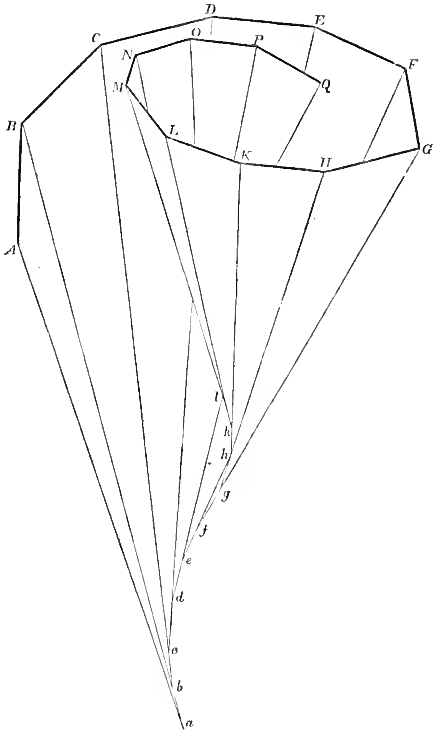
Developable Surfaces.

440. *Explanation of the development of developable surfaces into a plane.*

Let Aa, Bb, Cc, ... be a series of straight lines taken in order, according to any proposed law, so as to satisfy the condition that each intersects the preceding, viz. in the points a, b, c, ...

Since Aa, Bb intersect in a, they lie in the same plane, similarly the successive pairs of lines Bb and Cc, Cc and Dd, &c. lie in one plane; thus, a polygonal surface is formed by the successive plane elements AaB, BbC, &c.

This surface may be developed into one plane by turning the face AaB about Bb, until it forms a continuation of the plane BbC, and again turning the two, now forming one face, about Cc until the three AaB, BbC, CcD are in one plane, and so on; the whole surface may, therefore, be developed into one plane without tearing or rumpling; the same being true, however near the lines Aa, Bb, ... are taken, will be true in the limit, when the surface will become what we have defined as a developable surface, this name being derived from the property just proved.



Edge of Regression.

441. The polygon *abcd*, ... whose sides are in the direction of the lines *Bb*, *Cc*, ... becomes in the limit a curve, generally of double curvature, which is called the *Edge of Regression*, from the fact that the surface bends back at this curve so as to be of a cuspidal form. Every generating line of the system is a tangent to the edge of regression, which is therefore the envelope of all the generating lines.

In the case of a cylinder, the edge of regression is at an infinite distance.

For a practical construction of a developable surface having a given edge of regression, see Thompson and Tait, *Nat. Phil.*, Art. 149.

442. *To find the general nature of the intersection of a tangent plane to a developable surface with the surface.*

The plane containing the element DdE of the surface represented by the figure evidently becomes in the limit a tangent plane to the developable surface at any point D in the generating line Dd, since it contains two tangent lines, viz. Dd and the limiting position of a line joining such points as D and E, which ultimately coincide; and again, supposing DdE in the plane of the paper, Ff meets this plane in e, Ggf meets it in some point f', Hhg in g', &c., and similarly for Cc, Bb, ... on the other side.

The complete intersection of the surface and tangent plane is therefore the double line formed by the coincidence of Dd, Ee, and the limit of the polygon $a'b'cdef'g'$ ... which is a curve touching the double line Dd at the edge of regression.

COR. *To find the nature of the contact of the edge of regression and the tangent plane.*

The plane containing the generating lines Dd, Ee contains the three angular points c, d, e of the polygon in the limit, therefore the tangent plane contains two consecutive elements of the edge of regression, and is, as will be seen later on, what is called the osculating plane at that point.

443. *The shortest line which joins two points on a developable surface is the curve, the osculating plane at every point of which contains the normal to the surface at that point.*

If the surface be developed into a plane, the shortest line must be developed into the straight line joining the two points. If on the polygonal surface in the figure on page 294, $ABCD...K$ be the polygon which in the limit becomes the shortest line joining A and K, since on development this becomes a straight line, two consecutive sides EF, FG must be inclined at equal

angles to line Ff. Hence a straight line, drawn through F perpendicular to the line Ff in the plane bisecting the angle between the planes EFf, GFf, will evidently lie in the plane EFG, and bisect the angle EFG. This line will be in the limit the normal to the surface, and the plane EFG will be the osculating plane of the curve $ABCD$ ... at the point F.

Therefore the shortest line is the curve, the osculating plane at every point of which contains the normal to the surface at that point.

Such a line is called a geodesic line of the surface, and it will be hereafter shewn, that the property enunciated for developable surfaces is true for geodesic lines on all surfaces.

If the geodesic line, joining two given points, be drawn on a right circular cone, the equation of the projection upon the base can be shewn to be

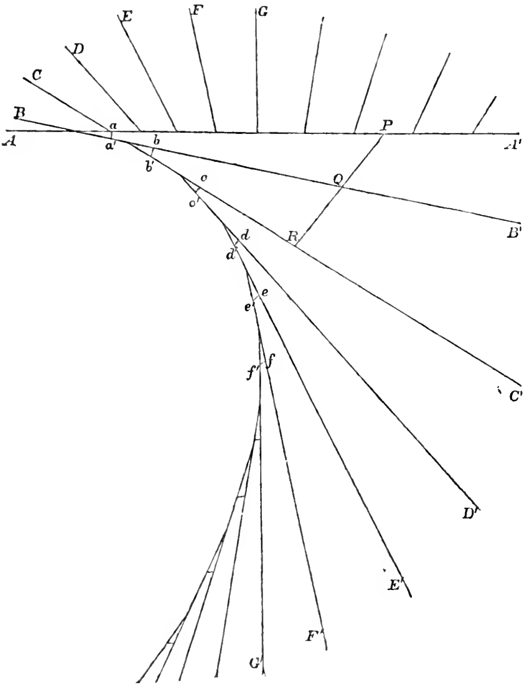
$$\frac{1}{r}\sin(\gamma \sin\alpha) = \frac{1}{b}\sin(\theta \sin\alpha) + \frac{1}{a}\sin\{(\gamma - \theta) \sin\alpha\},$$

a, b being the distances of the given points from the axis, γ the angle between these distances, and α the semi-vertical angle of the cone.

Skew Surfaces and Curves of greatest density.

444. Let AA', BB', CC', DD', &c. be straight lines drawn according to some fixed law, such that none intersects the next consecutive; let aa', bb', cc', dd', ... be the shortest distances. Suppose now that we take two of the generating lines as CC', DD', and imagine DD' twisted about c' so as to be parallel to CC', and united with it by means of an uniform elastic membrane: if now DD' be returned to its original position, the portion of the membrane near cc' being unstretched, will be denser than any other portion. If the same process be adopted for every line, the series of membranes will generate a surface which will ultimately, as the lines approach nearer to one another, become a *skew* or twisted surface.

The curve which is the limit of the polygon formed by joining a, b, c, d, ... at which the imagined membranes would have the greatest density, is called *the curve of greatest density;* it is also called the *line of striction.*



It may be observed, that the shortest distances between the consecutive generating lines of a scroll are not generally elements of the line of striction.

445. *To explain the nature of the contact of a tangent plane to a skew surface at any point.*

Let P be any point of a skew surface, AA' the generating line passing through P, suppose a plane to be drawn through P containing BB' the next consecutive position of the generating line, this plane will intersect the third line CC' in some point R,

and, if PR be joined, it will meet BB' in Q; PR will therefore be a tangent line at P having a contact of the second order at least, so that, if the surface were of the second order, it would lie entirely in the surface. The tangent plane at P is the plane containing AA' and PR; R will change its position for any change of position of P, thus the tangent plane at any point in AA' will always contain AA', but it will move about AA' through all positions, as the point of contact moves along AA'.

The tangent plane, therefore, at any point of a skew surface contains the generating line and some other curve which must be a straight line in the case of a surface of the second degree.

446. *To shew that the equation of the tangent plane to a developable surface contains only one parameter.*

Since the general equations of a straight line involve four arbitrary constants, we must, in order to generate any ruled surface, have three relations connecting the constants, so that it may be possible, between these equations and the two equations of the generating line, to eliminate the four constants, and thus obtain the equation of the surface which is the locus of all the straight lines. In developable surfaces the generating straight lines are such that any two consecutive ones intersect, and the plane containing them is ultimately a tangent plane to the surface. The equation of this plane will then involve the four parameters, and by means of the three relations we may eliminate three, so that the general equation of the tangent plane to a developable surface will involve only one parameter, and we may write it in the form

$$z = \alpha x + \phi(\alpha) y + \psi(\alpha),$$

α being the parameter, and $\phi(\alpha)$, $\psi(\alpha)$ functions of that parameter, given in any particular case.

Singular Tangent Plane.

447. DEF. A *singular tangent plane* is a plane which, instead of touching a surface in any finite number of points, touches along the whole of a curve line.

If the curve of intersection of any plane with the surface be composed, in part at least, of two or more coincident lines, the other part being made up of simple curves, either the plane will be a tangent plane to the surface at every point of such a multiple curve, or it will contain a multiple line of the surface, such as would be generated by the rotation of a cross round any fixed line not passing through the angle of the cross.

Conversely, if a tangent plane touch along a curve line on the surface, this curve line will be a multiple line on the tangent plane.

Thus, in the case of the anchor ring (Art. 435), the plane which touches the ring along a curve has for its curve of intersection the two circles coincident in LKH; also the tangent plane to a cone contains two generating lines which ultimately coincide, and is therefore a tangent plane at every point of the generating line which it contains; any more general developable surface is an example of the case of a tangent plane which contains a double line, at every point of which it is a tangent, combined, as shewn in Art. 442, with another simple curve.

A surface of the fourth degree admits of the case of a double conic, as in the example of the anchor ring, or of a quadruple straight line, as when it is made up of two cones touching along a generating line.

A surface of the fifth degree might be composed of one of the third degree and one of the second, in which case a tangent plane might meet the former in a triple and the latter in a double straight line.

448. *To find the condition that a tangent plane may be singular.*

Since a line, drawn in any direction in the tangent plane through any point of the double curve in which the tangent plane touches the surface, will contain two coincident points, but if it be drawn in the direction of the two coincident tangents to the curve of contact it will contain four coincident points, we have to express that at every point of the double curve there are two coincident tangents, and that a line in their direction contains four coincident points; and we may observe that

these tangents are what have been called inflexional tangents (Art. 432).

Since the two inflexional tangents coincide, their direction is given by the equations

$$\frac{u\lambda + w'\mu + v'\nu}{U} = \frac{w'\lambda + v\mu + u'\nu}{V} = \frac{v'\lambda + u'\mu + w\nu}{W},$$

$$\text{and } \lambda U + \mu V + \nu W = 0.$$

The condition that these equations shall hold is

$$\begin{vmatrix} u, & w', & v', & U \\ w', & v, & u', & V \\ v', & u', & w, & W \\ U, & V, & W, & 0 \end{vmatrix} = 0,$$

and the condition that a fourth point may become coincident is that for the values of $\lambda : \mu : \nu$ given by the above equations

$$\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy} + \nu \frac{d}{dz}\right)^3 F = 0\,;$$

the further conditions when the curve has a higher degree of of multiplicity may be easily obtained.

449. The conditions of the existence of a singular tangent plane may also be found, by considering that the point of contact, determined by the equations of Art. 425, may be any point of a curve line, and its coordinates are therefore indeterminate.

450. For a surface given by the unsymmetrical equation $\zeta = f(\xi, \eta)$, the equation of a tangent plane at any point (x, y, z) is

$$\zeta - z = p(\xi - x) + q(\eta - y)\,;$$

a tangent line whose equations are

$$\frac{\xi - x}{\lambda} = \frac{\eta - y}{\mu} = \frac{\zeta - z}{\nu} = \rho$$

meets the surface in points for which ρ is given by

$$\nu\rho = (p\lambda + q\mu)\rho + \tfrac{1}{2}\rho^2(r\lambda^2 + 2s\lambda\mu + t\mu^2)$$

$$+ \tfrac{1}{6}\rho^3\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy}\right)^3 z + \ldots \qquad (1)$$

If the tangent plane be singular, for all the points in which it meets the surface, $\nu = p\lambda + q\mu$, and for all the points of the double curve four values of ρ are zero, and the two inflexional tangents coincide;

$$\therefore\ r\lambda^2 + 2s\lambda\mu + t\mu^2 = 0 \text{ and } \left(\lambda \frac{d}{dx} + \mu \frac{d}{dy}\right)^3 z = 0,$$

and the former has equal roots, therefore $rt = s^2$, and either $r\lambda + s\mu = 0$ or $s\lambda + t\mu = 0$.

451. Every tangent plane to a developable surface is a singular tangent plane, since it contains two consecutive generating lines, hence its curve of intersection with the surface consists of two coincident straight lines, and, as shewn in Art. 442, a single curve line. The analytical conditions of singularity are satisfied, since, if λ, μ, ν be the direction-cosines of the double line, which lies entirely in the surface, the coefficients of *all* the powers of ρ will vanish, and $rt = s^2$ in consequence of the coincidence of the two lines.

At any point of the single curve, the values of p, q being p', q', the direction cosines λ, μ, ν of the tangent are given by $\nu = \lambda p + \mu q$ and $\nu = \lambda p' + mq'$, which are independent equations, since p', p and q', q are generally unequal.

452. We have selected the following illustrations of the points which have been considered in this chapter, and we call attention especially to those relating to cubic surfaces and the wave surface as of intrinsic importance.

453. *Tangent plane to an anchor ring.*

Let the plane containing the centres of the generating circles be taken for the plane of xy, and the axis of rotation for the axis of z; and let r be the distance of any point (x, y, z) from the axis, c that of the centre of the generating circle, a its radius; then $r^2 = x^2 + y^2$ and $z^2 + (r-c)^2 = a^2$; the equation of the anchor ring is

$$(\xi^2 + \eta^2 + \zeta^2 + c^2 - a^2)^2 - 4c^2(\xi^2 + \eta^2) = 0,$$

that of the tangent plane at a point (x, y, z) is

$$x(r-c)(\xi - x) + y(r-c)(\eta - y) + zr(\zeta - z) = 0,$$

$$\text{or}\ \ (r-c)(x\xi + y\eta) + rz\zeta = r^2(r-c) + rz^2 = r\{a^2 + c(r-c)\}.$$

To find the curve of intersection of the surface with a tangent plane which passes through the centre.

Suppose that it passes through axis of y, and is inclined at angle α to that of x, so that $a = c\sin\alpha$; and at any point of the curve of intersection $r = c - a\cos\theta$, $z = a\sin\theta$, $x = z\cot\alpha$;

$$\begin{aligned}\therefore\ y^2 = r^2 - x^2 &= c^2 - 2ac\cos\theta + a^2\cos^2\theta - a^2\cot^2\alpha\sin^2\theta \\ &= c^2 - 2ac\cos\theta + a^2\cos^2\theta - (c^2 - a^2)\sin^2\theta \\ &= (c\cos\theta - a)^2;\end{aligned}$$

$$\therefore\ (y \pm a)^2 = c^2\cos^2\theta,$$

$$\text{and}\ \ x^2 + z^2 = c^2\sin^2\theta;$$

$$\therefore\ x^2 + (y \pm a)^2 + z^2 = c^2;$$

hence the curve is two circles which intersect in the points of contact, forming two double points.

To find the form of the curve EAF in the figure of the ring.

The equation of the tangent plane is $\xi = c - a$, and the form of the curve of intersection is given by the equation

$$\{\eta^2 + \zeta^2 + 2c(c-a)\}^2 = 4c^2\{\eta^2 + (c-a)^2\},$$

$$\text{or}\ \ (\eta^2 + \zeta^2)^2 - 4ac\eta^2 + 4c(c-a)\zeta^2 = 0.$$

When $c = 2a$, the curve is the lemniscate of Bernoulli.

454. *Tangent plane and normal to a Helicoid.*

Def. The Helicoid is a scroll generated by the motion of a straight line which intersects at right angles a fixed axis, about which it twists with an angular velocity which varies as the velocity of the point of intersection with the axis.

If the axis be taken for the axis of z, and that of x be one position of the generating line, the equation of the surface generated will be

$$\zeta = c\tan^{-1}\frac{\eta}{\xi},$$

and the tangent plane at a point (x, y, z) will be

$$\zeta - z = -\frac{cy}{x^2 + y^2}(\xi - x) + \frac{cx}{x^2 + y^2}(\eta - y),$$

$$\text{or } (x^2+y^2)(\zeta-z)=c(x\eta-y\xi);$$

at the point $(x, 0, 0)$, the equation becomes $x\zeta=c\eta$, hence the tangent of the angle which the tangent plane at any point makes with the axis varies as the distance of the point from the axis.

The equations of the normal at (x, y, z) are

$$\frac{\xi-x}{y}=\frac{\eta-y}{-x}=\frac{c(\zeta-z)}{x^2+y^2};$$

and for the normal at $(x, 0, 0)$, $\xi=x$, $x\eta+c\zeta=0$, hence the locus of the normals at points taken along a generating line is an hyperbolic paraboloid; which is true for any scroll.

455. *To find the singularities of the surface whose equation is*

$$(z^2+2x^2+2y^2)^2-(x^2+y^2)(x^2+y^2+1)^2=0.$$

We consider this surface as represented by the given equation, in order to illustrate the general methods given in Arts. 427 and 448, for discussing singular points and planes; but the student will see clearly the results to which we shall be led, if he first trace the plane curve whose equation is $y^2=x(1-x)^2$,* and then imagine the form of the surface which would be generated by its revolution round the axis of y, which it is easily seen is the surface proposed.

To find a singular point we have, writing r^2 for x^2+y^2,

$$U=2x(4z^2-1+4r^2-3r^4)=0,$$
$$V=2y(4z^2-1+4r^2-3r^4)=0,$$
$$W=4z(z^2+2r^2)=0.$$

The systems of values of x, y, z which simultaneously satisfy these equations, and that of the surface are $z=0$, and either (i) $x=0$, $y=0$, or (ii) $r^2=1$; (i) shews that the origin is a singular point—it will be found that the tangent cone of Art. 428 becomes an infinitely slender cylinder or cone, given by $\lambda^2+\mu^2=0$; (ii) gives a circle of singular points—the conical tangent at any point $(x, y, 0)$ of this circle becomes the two tangent planes $(x\xi+y\eta-1)^2=\zeta^2$.

* Frost's *Curve Tracing*, Plate II., Fig. 8.

To find a singular tangent plane we have, by Art. 448, the equations

$$2(\lambda x + \mu y)(4z^2 - 1 + 4r^2 - 3r^4) + 4z(z^2 + 2r^2)\nu = 0,$$

and $2(\lambda^2 + \mu^2)(4z^2 - 1 + 4r^2 - 3r^4) + 4\nu^2(3z^2 + 2r^2)$

$$+ 32z(\lambda x + \mu y)\nu + 8(\lambda x + \mu y)^2(2 - 3r^2) = 0;$$

there will be two coincident tangents if

$$\nu = 0 \text{ and } 4z^2 - 1 + 4r^2 - 3r^4 = 0,$$

also by the equation of the surface $(z^2 + 2r^2)^2 - r^2(1 + r^2)^2 = 0$, the only solutions of these equations are $z^2 = 0$, $r^2 = 1$, and $z^2 = \frac{4}{27}$, $r^2 = \frac{1}{9}$, the first solution gives no tangent plane, but two cones intersecting in a circle, any generating line of either of which is a tangent line; the second solution gives two tangent planes $z = \pm \frac{2}{9}\sqrt{3}$, each of which is a singular tangent plane touching along a circle $x^2 + y^2 = \frac{1}{9}$, the direction of the tangent to which is given by $\lambda x + \mu y = 0$ and $\nu = 0$, the remaining part of the curve of intersection is a single circle of radius $\frac{4}{3}$.

In this case the condition of four points being coincident is $\frac{1}{\underline{|3}}\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy}\right)^3 F = 0$, which becomes

$$\tfrac{20}{3}(\lambda x + \mu y)(\lambda^2 + \mu^2) - 8(\lambda x + \mu y)^3 = 0,$$

it is therefore satisfied by $\lambda x + \mu y = 0$.

Wave Surface.

456. The equation of the Wave Surface may be written in either of the forms

$$\frac{x^2}{r^2 - a^2} + \frac{y^2}{r^2 - b^2} + \frac{z^2}{r^2 - c^2} = 1,$$

$$\text{or} \quad \frac{a^2x^2}{r^2 - a^2} + \frac{b^2y^2}{r^2 - b^2} + \frac{c^2z^2}{r^2 - c^2} = 0,$$

where $r^2 = x^2 + y^2 + z^2$; and we shall suppose $a > b > c$.

The existence of singular tangent planes to this surface is of great importance in explaining a peculiarity in the transmission of light through a biaxial crystal.

In order to shew that such planes exist, we shall employ the method of Art. 449.

457. *To find the point of contact of a tangent plane whose equation is $lx + my + nz = p$, and the relation between l, m, n and p.*

The equations for determining the point of contact are

$$\frac{U}{l} = \frac{V}{m} = \frac{W}{n} = \frac{xU + yV + zW}{p} = -2\sigma \text{ suppose,}$$

where $$U = \frac{2x}{r^2 - a^2} - 2xP,$$

$$V = \frac{2y}{r^2 - b^2} - 2yP,$$

$$W = \frac{2z}{r^2 - c^2} - 2zP,$$

$$P = \frac{x^2}{(r^2 - a^2)^2} + \frac{y^2}{(r^2 - b^2)^2} + \frac{z^2}{(r^2 - c^2)^2};$$

$$\therefore\ xU + yV + zW = 2 - 2r^2P = -2\sigma p,$$

and $$U^2 + V^2 + W^2 = 4(P - 2P + r^2P^2) = 4\sigma^2;$$

$$\therefore\ P\sigma p = \sigma^2,\ \ (r^2 - p^2)P = 1,\ \text{ and }\ (r^2 - p^2)\sigma = p,$$

and $U = -2l\sigma$; $$\therefore\ x\left(\frac{1}{r^2 - a^2} - \frac{1}{r^2 - p^2}\right) = -\frac{lp}{r^2 - p^2};$$

$$\therefore\ \frac{x}{r^2 - a^2} = \frac{lp}{p^2 - a^2},\ \text{\&c.}; \tag{1}$$

hence, $$x = lp\left(\frac{r^2 - p^2}{p^2 - a^2} + 1\right),\ \text{\&c.},$$

and multiplying by l, m, n, and adding, we obtain

$$\frac{l^2}{p^2 - a^2} + \frac{m^2}{p^2 - b^2} + \frac{n^2}{p^2 - c^2} = 0. \tag{2}$$

Also by squaring and adding, and observing (2),

$$r^2 = x^2 + y^2 + z^2 = p^2(r^2 - p^2)^2\left\{\frac{l^2}{(p^2 - a^2)^2} + \frac{m^2}{(p^2 - b^2)^2} + \frac{n^2}{(p^2 - c^2)^2}\right\} + p^2;$$

$$\therefore\ \frac{1}{r^2 - p^2} = p^2\left\{\frac{l^2}{(p^2 - a^2)^2} + \frac{m^2}{(p^2 - b^2)^2} + \frac{n^2}{(p^2 - c^2)^2}\right\}. \tag{3}$$

The equations (1) and (3) give the values of x, y, z at the point of contact, and (2) is the required relation between the constants.

If α, β, γ be the Boothian coordinates of the tangent plane, viz. $\frac{l}{p}$, $\frac{m}{p}$, $\frac{n}{p}$, and a', b', c' be written for $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, the tangential equation of the surface will be

$$\frac{a'^2\alpha^2}{\rho^2 - a'^2} + \frac{b'^2\beta^2}{\rho^2 - b'^2} + \frac{c'^2\gamma^2}{\rho^2 - c'^2} = 0,$$

where $\rho^2 = \alpha^2 + \beta^2 + \gamma^2$, an equation of the same form as the Cartesian.

458. *To find a singular tangent plane of the wave surface.*

The point of contact in this case being any point in a curve, the coordinates must be indeterminate; now y will be of the form $\frac{0}{0}$ if $p = b$ and $m = 0$, and these values also make r^2 indeterminate, and therefore x and z.

And since, by (2) and (3),

$$\frac{p^2 - b^2}{r^2 - p^2} = p^2 \left\{ \frac{l^2 (a^2 - b^2)}{(p^2 - a^2)^2} - \frac{n^2 (b^2 - c^2)}{(p^2 - c^2)^2} \right\},$$

$$\text{we have } \frac{l^2}{a^2 - b^2} = \frac{n^2}{b^2 - c^2} = \frac{1}{a^2 - c^2}.$$

The curve of intersection is a circle given by the plane $lx + nz = p$, and either of the spheres

$$r^2 + \frac{a^2 - b^2}{lb} x = a^2,$$

$$\text{or } r^2 - \frac{b^2 - c^2}{nb} z = c^2,$$

$p = a$, $l = 0$ and $p = c$, $n = 0$ give imaginary planes, hence there are four real singular tangent planes.

459. *To find the singular points and the corresponding normal cones.*

The singular points may be found by investigating for what definite points of contact l, m, n can be indeterminate, we shall thus obtain

$$y = 0, \quad r = b, \quad \text{and} \quad \frac{a^2x^2}{a^2 - b^2} = \frac{c^2z^2}{b^2 - c^2} = \frac{a^2c^2}{a^2 - c^2}, \qquad (4)$$

which determine four singular points.

By equations (1) of Art 457,

$$-\frac{a^2-b^2}{x}\,l=p-\frac{a^2}{p}\quad\text{and}\quad\frac{b^2-c^2}{z}\,n=p-\frac{c^2}{p};$$

$$\therefore\ \frac{a^2-b^2}{x}\,l+\frac{b^2-c^2}{z}\,n=\frac{a^2-c^2}{p};$$

$$\therefore\ (a^2xl+c^2zn)\,(lx+nz)=a^2c^2,\ \text{by (4)},$$

$$\text{or}\ a^2x^2l^2+c^2z^2n^2+(a^2+c^2)\,xzln=a^2c^2,$$

$$\therefore\ (a^2-b^2)\,l^2+(b^2-c^2)\,n^2+\frac{(a^2+c^2)}{ac}\sqrt{(a^2-b^2)}\,\sqrt{(b^2-c^2)}\,ln$$
$$=(a^2-c^2)\,(l^2+m^2+n^2),$$

$$\text{or}\ (b^2-c^2)\,l^2+(a^2-c^2)\,m^2+(a^2-b^2)\,n^2-\frac{a^2+c^2}{ac}\sqrt{(a^2-b^2)}\,\sqrt{(b^2-c^2)}\,ln=0;$$

which gives the equation of the normal cone at the singular point.

Cubic Surfaces.

460. *On every surface of the third degree there are* 27 *straight lines and* 45 *triple tangent planes, real or imaginary.*

This theorem was first discovered by Cayley.*

An arbitrary straight line intersects a cubic surface in three points, given by an equation of the form

$$u+Du\,.\,r+\tfrac{1}{2}D^2u\,.\,r^2+\tfrac{1}{6}D^3u\,.\,r^3=0.$$

Now the four constants in the equations of a line may be chosen so as to satisfy the equations $u=0$, $Du=0$, $D^2u=0$, $D^3u=0$, and, since the above equation will then be satisfied by all values of r, all straight lines having such constants will lie entirely in the surface; and the number of such straight lines will clearly be limited, speaking generally, although in particular cases, as in that of a cylindrical surface, it may be infinite.

If a plane be drawn in any direction through such a straight line, its line of intersection with the surface will be composed of that straight line and a conic forming a group of the third degree; and the two double points in which the straight line

* *Cambridge and Dublin Mathematical Journal*, vol. IV.

intersects the conic are two points of the surface at which the plane is a tangent plane to the cubic (Art. 434).

Now, there will be five positions of the plane for which the conic will become two straight lines.

For, if the axis of x be a line which lies entirely in the surface, the equation of the surface will be of the form

$$yu_2 + zv_2 = 0,$$

where u_2, v_2 are quadric functions; and if the surface be cut by a plane whose equation is $\frac{y}{\mu} = \frac{z}{\nu} = r$, the conic, which is part of the line of intersection, will have for its equation $\mu u'_2 + \nu v'_2 = 0$, where u'_2, v'_2 have each the form

$$a_2 r^2 + b_0 x^2 + c_0 + 2d_0 x + 2e_1 r + 2f_1 rx = 0,$$

in which a_2, e_1, f_1 are homogeneous functions of μ and ν of the degrees denoted by the suffixes.

Hence the equation of the conic will be

$$\alpha_3 r^2 + \beta_1 x^2 + \gamma_1 + 2\delta_1 x + 2\varepsilon_2 r + 2\zeta_2 rx = 0;$$

it will therefore become two straight lines if

$$\alpha_3\beta_1\gamma_1 + 2\delta_1\varepsilon_2\zeta_2 - \alpha_3\delta_1{}^2 - \beta_1\varepsilon_2{}^2 - \gamma_1\zeta_2{}^2 = 0,$$

which gives five values of the ratio $\lambda : \mu$.

In each of the five particular positions of the plane the complete intersection is three straight lines, which give three double points, and the plane is a triple tangent plane touching the surface at each of these double points.

Through each of the three straight lines in a triple tangent plane four other triple tangent planes besides the one considered can be drawn, giving rise to 12 new triple tangent planes and 24 new straight lines, making in all 27; and the surface cannot contain any but these 27 lines, for the point in which any line on the surface meets a triple tangent plane ABC must lie on one of the three lines AB, BC, CA, which form the complete intersection of ABC with the surface, and the plane which passes through the new line and AB, supposing this to be the line which it cuts, must contain a third line, and, therefore, must be one of the five triple tangent planes drawn through AB; the line considered must therefore be one of the 27 lines.

Five triple tangent planes can be drawn through each of the 27 lines, which would make 5×27 planes in all; but since each plane contains three of the lines, we have in obtaining this number reckoned each three times, hence the number of triple tangent planes is 45.

Line of Striction.

461. *To find the line of striction of a scroll.*

Let the equation of a generating line be

$$\eta = m\xi + \alpha, \quad \zeta = n\xi + \beta, \tag{1}$$

where the constants are functions of one parameter θ; the equations of a consecutive generator corresponding to a value $\theta + d\theta$ of the parameter are

$$\eta = (m + dm)\,\xi + \alpha + d\alpha, \quad \zeta = (n + dn)\,\xi + \beta + d\beta. \tag{2}$$

Let P be a point in the line of striction; PQ the shortest distance between (1) and (2); x, y, z and $x + \delta x$, $y + \delta y$, $z + \delta z$ the coordinates of P and Q.

Since PQ is perpendicular to both generators,

$$\delta x + m\delta y + n\delta z = 0,$$

$$\text{and } \delta x + (m + dm)\,\delta y + (n + dn)\,\delta z = 0;$$

$$\therefore\ dm\delta y + dn\delta z = 0.$$

Also, by the equations (1) and (2),

$$\delta y - m\delta x = xdm + d\alpha,$$

$$\delta z - n\delta x = xdn + d\beta;$$

$$\therefore\ \frac{\delta y}{dn} = \frac{\delta z}{-dm} = \frac{\delta x}{ndm - mdn}$$

$$= \frac{\delta y - m\delta x}{(1 + m^2)\,dn - mndm} = \frac{\delta z - n\delta x}{-(1 + n^2)\,dm + mndn};$$

$$\therefore\ (xdm + d\alpha)\,\{(1 + n^2)\,dm - mndn\}$$

$$+ (xdn + d\beta)\,\{(1 + m^2)\,dn - mndm\} = 0.$$

If the parameter θ be eliminated between this equation and the equations

$$y = mx + \alpha, \quad z = nx + \beta,$$

we shall obtain two equations which will be those of the line of striction.

462. *Line of striction of an hyperboloid of one sheet.*

For the hyperboloid $\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x^2}{a^2} + 1$, the equations of a generator being

$$\frac{y}{b} = \frac{x}{a}\cos\theta + \sin\theta,\ \frac{z}{c} = \frac{x}{a}\sin\theta - \cos\theta,$$

$$xdm + d\alpha = b\left(-\frac{x}{a}\sin\theta + \cos\theta\right)d\theta = -\frac{bz}{c}\,d\theta,$$

$$xdm + d\beta = c\left(\ \ \frac{x}{a}\cos\theta + \sin\theta\right)d\theta = \ \ \frac{cy}{b}\,d\theta,$$

$$(1 + n^2)\,dm - mndn = -\frac{bc^2}{a}\left(\frac{1}{c^2} + \frac{1}{a^2}\right)\sin\theta\,d\theta,$$

$$(1 + m^2)\,dn - mndm = \frac{cb^2}{a}\left(\frac{1}{b^2} + \frac{1}{a^2}\right)\cos\theta\,d\theta;$$

$$\therefore\ \left(\frac{1}{c^2} + \frac{1}{a^2}\right)\frac{z}{c}\sin\theta + \left(\frac{1}{b^2} + \frac{1}{a^2}\right)\frac{y}{b}\cos\theta = 0;$$

$$\text{also}\ \left(\frac{y}{b} + \frac{xz}{ac}\right)\cos\theta + \left(\frac{z}{c} - \frac{xy}{ab}\right)\sin\theta = 0,$$

$$\text{whence}\ \left(\frac{1}{c^2} + \frac{1}{a^2}\right)\left(\frac{y}{b} + \frac{xz}{ac}\right)\frac{z}{c} - \left(\frac{1}{b^2} + \frac{1}{a^2}\right)\left(\frac{z}{c} - \frac{xy}{ab}\right)\frac{y}{b} = 0;$$

$$\therefore\ \left(\frac{1}{c^2} - \frac{1}{b^2}\right)\frac{yz}{bc} + \frac{x}{a}\left(\frac{z^2}{c^4} + \frac{y^2}{b^4} + \frac{x^2}{a^4} + \frac{1}{a^2}\right) = 0;$$

the intersection of this surface with the hyperboloid gives the line of striction for one set of generators.

Polar Equation.

463. *To find the polar equation of the tangent plane to a surface at a given point.*

Let the equation of the surface be $\frac{1}{r'} = u' = f(\theta', \phi')$, and let u, θ, ϕ be the coordinates of the point of contact of the tangent plane.

The equation of the tangent plane is of the form

$$pu' = \cos\alpha \cos\theta' + \sin\alpha \sin\theta' \cos(\phi' - \beta), \text{ (Art. 76)},$$

and the constants p, α, and β are to be determined from the consideration that the tangent plane contains not only the point of contact but adjacent points which have moved up to and ultimately coincided with that point.

Hence the values of $\frac{du}{d\theta}$ and $\frac{du}{d\phi}$ at the point of contact are the same for both tangent plane and surface, let v, w be those values;

$$\begin{aligned} \therefore\ pu &= \quad \cos\alpha \cos\theta + \sin\alpha \sin\theta \cos(\phi - \beta), \\ pv &= -\cos\alpha \sin\theta + \sin\alpha \cos\theta \cos(\phi - \beta), \\ pw &= \qquad\qquad\quad - \sin\alpha \sin\theta \sin(\phi - \beta); \end{aligned}$$

$$\therefore\ p(u \sin\theta + v \cos\theta) = \sin\alpha \cos(\phi - \beta),$$

$$p(u \cos\theta - v \sin\theta) = \cos\alpha;$$

the last three equations give readily the values of the constants; and the equation of the tangent plane becomes

$$\begin{aligned} u' = (u \cos\theta - v \sin\theta) \cos\theta' \\ + (u \sin\theta + v \cos\theta) \cos(\phi' - \phi) \sin\theta' \\ + w \operatorname{cosec}\theta \sin(\phi' - \phi) \sin\theta'. \end{aligned}$$

This equation can also be written in the form

$$\begin{aligned} \frac{r^2}{r'} = \frac{d}{d\theta} [r \{\sin\theta \cos\theta' - \cos\theta \sin\theta' \cos(\phi' - \phi)\}] \\ - \frac{dr}{d\phi} \operatorname{cosec}\theta \sin\theta' \sin(\phi' - \phi). \end{aligned}$$

464. *To find the perpendicular distance from the pole upon the tangent plane.*

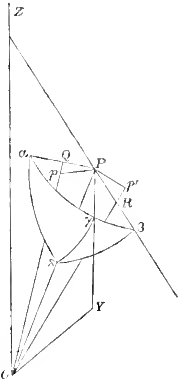
This may be obtained from the first three equations of the last article by squaring and adding, whence

$$p^2 (u^2 + v^2 + w^2 \operatorname{cosec}^2\theta) = 1,$$

$$\text{or, } \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 + \left(\frac{du}{d\phi}\right)^2 \operatorname{cosec}^2\theta.$$

465. We may arrive at the above result by the following process, which serves to shew the geometrical signification of the partial differential coefficients, and will be useful as an exercise.

Let P be the point of contact, PR a tangent line passing through OZ, and PQ a tangent line in the plane through OP perpendicular to the plane POZ; take R and Q points very



near to P, and in OQ, OR take Op, Op' each equal to OP; then $Pp = r \sin\theta d\phi$ and $Pp' = rd\theta$ ultimately, and Qp, $-Rp'$ are respectively the values of dr due to changes of θ and ϕ, considering the other constant,

$$\therefore \frac{dr}{rd\theta} = -\frac{Rp'}{Pp'} = -\cot OPR,$$

$$\text{and } \frac{dr}{r\sin\theta d\phi} = \frac{Qp}{Pp} = -\cot OPQ.$$

Draw OY perpendicular to the tangent plane QPR, and on a sphere, whose centre is P, let $\alpha\delta\beta$ be a spherical triangle with its angular points in PQ, PO, PR, join $\delta\gamma$, γ being the intersection of PY and $\alpha\beta$, then $\delta\gamma$ is perpendicular to $\alpha\beta$, and $\alpha\delta\beta$ is a right angle. Hence

$$\cot\alpha\delta = \cot\delta\gamma\cos\alpha\delta\gamma, \text{ and } \cot\beta\delta = \cot\delta\gamma\sin\alpha\delta\gamma;$$

$$\therefore\ \cot^2\alpha\delta + \cot^2\beta\delta = \cot^2\delta\gamma = \frac{r^2 - p^2}{p^2};$$

$$\therefore\ \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 \operatorname{cosec}^2\theta.$$

Asymptotes.

466. Def. An *asymptote* to a surface is a straight line which meets the surface in two points, at least, at an infinite distance, while the line itself remains at a finite distance.

An *asymptotic plane* is a tangent plane whose point of contact is at an infinite distance, the plane itself being at a finite distance.

An *asymptotic surface* is a surface which is enveloped by all the asymptotic planes to the surface.

467. *General considerations on asymptotes.*

If we imagine any tangent plane to a surface, and consider the result of supposing its point of contact to be at an infinite distance, we shall be led to the following conclusions:

Since the plane at infinity intersects the surface in a curve, real or imaginary, there are generally an infinite number of directions in which a point of contact may be supposed to move off to infinity; to each of these directions will correspond an asymptotic plane.

Each asymptotic plane is the locus of all the corresponding asymptotes, and these asymptotes will all be parallel, since they pass through the same point at infinity at which they are tangents.

Since there are two tangents in every tangent plane at an ordinary point which pass through three consecutive points, viz. the tangents to the curve of intersection at the point of contact, there are in each asymptotic plane two corresponding inflexional asymptotes which pass through three points at an infinite distance.

Since any plane which passes through an inflexional tangent intersects the surface in a curve which has a point of inflexion at the point of contact of such a tangent, the curve of inter-

section of the surface and any plane drawn through an inflexional asymptote has a point of inflexion at an infinite distance.

468. The peculiarities which arise in the case of singular points at an infinite distance can be examined without much difficulty by a comparison with what takes place at a finite distance.

If, for example, there be a double point at infinity, in the place of the conical tangent at a finite distance, there will be a cylinder of the second degree formed by the asymptotes which correspond to the direction in which the double point lies.

Of the generating lines of this asymptotic cylinder there are six which meet the surface in four points at infinity.

The curve of intersection with any plane parallel to these generating lines has a double point at infinity.

The curve of intersection with any tangent plane to the cylindrical asymptote has a cusp at infinity.

469. *To find the asymptotes to a given surface.*

Let $F \equiv F(\xi, \eta, \zeta) = 0$ be the equation of the given surface, (x, y, z) any point in an asymptote, $\dfrac{\xi - x}{\lambda} = \dfrac{\eta - y}{\mu} = \dfrac{\zeta - z}{\nu} = r$ its equations; and let $F(\lambda, \mu, \nu)$ be arranged in a series of homogeneous functions of the degrees n, $n-1$, ..., so that

$$F(\lambda, \mu, \nu) \equiv \phi_n + \phi_{n-1} + \ldots + \phi_1 + c.$$

The points in which the asymptote meets the surface are given by the equation

$$F(x + \lambda r,\ y + \mu r,\ z + \nu r) = 0,$$

or, if D denote the operation $x\dfrac{d}{d\lambda} + y\dfrac{d}{d\mu} + z\dfrac{d}{d\nu}$,

$$r^n\phi_n + r^{n-1}(D\phi_n + \phi_{n-1}) + r^{n-2}(\tfrac{1}{2}D^2\phi_n + D\phi_{n-1} + \phi_{n-2}) + \ldots = 0.$$

Now for a simple asymptote two roots are infinite;

$$\therefore\ \phi_n = 0 \tag{1}$$

$$\text{and}\ \ D\phi_n + \phi_{n-1} = 0. \tag{2}$$

The first equation shews that all asymptotes are parallel to generating lines of the cone

$$F_n(\xi, \eta, \zeta) = 0, \tag{3}$$

where F_n consists of the terms of the n^{th} degree in F.

The second equation

$$x\frac{d\phi_n}{d\lambda} + y\frac{d\phi_n}{d\mu} + z\frac{d\phi_n}{d\nu} + \phi_{n-1} = 0$$

shews that all the asymptotes parallel to any generating line of the cone (3) lie in one plane, which is the asymptotic plane parallel to the tangent plane touching the cone along the generating line.

Again, corresponding to inflexional tangents in tangent planes at points at a finite distance, there are generally two asymptotes in each asymptotic plane which meet the surface in three points at an infinite distance, the condition of this is

$$\tfrac{1}{2}D^2\phi_n + D\phi_{n-1} + \phi_{n-2} = 0, \tag{4}$$

and the two inflexional asymptotes are the lines of intersection of the conicoid (4) with the plane (2).

It can be shewn that the conicoid and plane intersect in two parallel or coincident lines by proving that, if (x, y, z) be any point in which they intersect, a line drawn through this point in the direction (λ, μ, ν) lies entirely in both surfaces.

Write $x + \lambda r$ for x, &c., and Δ for the operation

$$\lambda\frac{d}{d\lambda} + \mu\frac{d}{d\mu} + \nu\frac{d}{d\nu},$$

λ, μ ν being considered constant in the differentiations.

$$(2) \text{ becomes } (D + r\Delta)\,\phi_n + \phi_{n-1} = r\Delta\phi_n = rn\phi_n,$$

$$(4) \text{ becomes } \tfrac{1}{2}(D + r\Delta)^2\,\phi_n + (D + r\Delta)\,\phi_{n-1} + \phi_{n-2}$$

$$= r\Delta D\phi_n + \tfrac{1}{2}r^2\Delta^2\phi_n + r\Delta\phi_{n-1}$$

$$= r(n-1)(D\phi_n + \phi_{n-1}) + \tfrac{1}{2}r^2 n(n-1)\,\phi_n = \tfrac{1}{2}r^2 n(n-1)\,\phi_n;$$

therefore, since $\phi_n = 0$, (2) and (4) are satisfied for all values of r.

470. Should the student be interested in the discrimination of the various singularities which may occur, he will find a guide

in two articles by Painvin,* who has nearly adopted our method of treatment, and has carefully followed out the consequences of supposing the conicoid (4) to have the various forms of which it is capable.

471. A singular asymptotic plane is one which touches the surface along a line at infinity, if considered as the limit of a tangent plane; and if considered as the locus of asymptotic lines, it is a plane such that lines drawn in *any* direction in it meet the surface in two points at an infinite distance.

The analytical conditions are obtained by considering that the equation $D\phi_n + \phi_{n-1} = 0$ must be independent of the values of λ, μ, ν.

Asymptotic Surfaces.

472. *To find the asymptotic surface of a given surface.*

The asymptotic surface being the surface enveloped by the asymptotic planes, which are tangent planes whose points of contact are at an infinite distance, is a developable surface circumscribing the surface along the curve of intersection with the plane at infinity.

The equation of an asymptotic plane is

$$P \equiv x\frac{d\phi_n}{d\lambda} + y\frac{d\phi_n}{d\mu} + z\frac{d\phi_n}{d\nu} + \phi_{n-1} = 0, \tag{2}$$

λ, μ, ν being connected by the equations

$$\phi_n = 0 \quad\text{and}\quad \lambda^2 + \mu^2 + \nu^2 = 1.$$

We shall write u, u'... for $\dfrac{d^2\phi_n}{d\lambda^2}$, $\dfrac{d^2\phi_n}{d\mu\,d\nu}$, ..., and U_n, U_{n-1}... for $\dfrac{d\phi_n}{d\lambda}$, $\dfrac{d\phi_{n-1}}{d\lambda}$, ...

Considering a consecutive position of the asymptotic plane, we have the equations

$$\frac{dP}{d\lambda}d\lambda + \frac{dP}{d\mu}d\mu + \frac{dP}{d\nu}d\nu = 0,$$

$$U_n\,d\lambda + V_n\,d\mu + W_n\,d\nu = 0,$$

$$\lambda\ d\lambda + \mu\ d\mu + \nu\ d\nu = 0;$$

* Crelle's *Journal*, vol. 65.

therefore, by arbitrary multipliers,

$$\frac{dP}{d\lambda} + AU_n + B\lambda = 0,$$

$$\frac{dP}{d\mu} + AV_n + B\mu = 0,$$

$$\frac{dP}{d\nu} + AW_n + B\nu = 0,$$

and, multiplying by λ, μ, ν,

$$(n-1)P + An\phi_n + B = 0, \quad \therefore \; B = 0,$$

$$\text{hence } \frac{\dfrac{dP}{d\lambda}}{U_n} = \frac{\dfrac{dP}{d\mu}}{V_n} = \frac{\dfrac{dP}{d\nu}}{W_n}. \tag{5}$$

These equations and (2) are equivalent to two distinct equations which are those of a generating line of the asymptotic surface; that of the surface itself is found by eliminating λ, μ, ν from $\phi_n = 0$ and these two equations, all being homogeneous in λ, μ, ν.

If $\phi_{n-1} \equiv 0$, since $(n-1)U_n = \lambda u + \mu w' + \nu v'$, and $\dfrac{dP}{d\lambda} = xu + yw' + zv'$, the equations (5) are reduced to $\dfrac{x}{\lambda} = \dfrac{y}{\mu} = \dfrac{z}{\nu}$, and the asymptotic surface becomes the cone $\phi_n = 0$, as in the case of $ax^2 + by^2 + cz^2 = 1$.

In the general case it is easily seen that the generating line passes through the centre of the conicoid which determines the position of the inflexional asymptotes, for which

$$\frac{dP}{d\lambda} = \frac{dP}{d\mu} = \frac{dP}{d\nu} = 0.$$

473. *To find the degree of the asymptotic surface.*

We shall find how many generating lines intersect an arbitrary straight line $\dfrac{x-\alpha}{l'} = \dfrac{y-\beta}{m'} = \dfrac{z-\gamma}{n'} = r$. If we equate to $(n-1)\rho$ each member of equations (5) of a generating line, the equations may be written

$$(x-\lambda\rho)\,u + (y-\mu\rho)\,w' + (z-\nu\rho)\,v' + U_{n-1} = 0,$$

$$(x-\lambda\rho)\,w' + (y-\mu\rho)\,v + (z-\nu\rho)\,u' + V_{n-1} = 0,$$

$$(x-\lambda\rho)\,v' + (y-\mu\rho)\,u' + (z-\nu\rho)\,w + W_{n-1} = 0,$$

$$\text{or, if } H \equiv \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix},$$

$$(x-\lambda\rho)\, H + \left(U_{n-1}\frac{d}{du} + V_{n-1}\frac{d}{dw'} + W_{n-1}\frac{d}{dv'}\right) H = 0\,;$$

therefore at the point of intersection

$$(\alpha + l'r)\, H - \lambda\rho H + \left(U_{n-1}\frac{d}{du} + \ldots\right) H = 0,$$

and similar equations; or, eliminating r and ρ,

$$\begin{vmatrix} \alpha H + \left(U_{n-1}\dfrac{d}{du} + \ldots\right) H, & \lambda, & l' \\ \beta H + \left(U_{n-1}\dfrac{d}{dw'} + \ldots\right) H, & \mu, & m' \\ \gamma H + \left(U_{n-1}\dfrac{d}{dv'} + \ldots\right) H, & \nu, & n' \end{vmatrix} = 0,$$

now the degrees of $\dfrac{dH}{du} \equiv vw - u'^2$ and U_{n-1} are $2(n-2)$ and $n-2$; the degree of the equation is therefore $3n-5$, and the number of values of λ, μ, ν which satisfy this equation, and $\phi_n = 0$ is $n(3n-5)$, which is the degree of the asymptotic surface.

474. Or we may proceed thus:

The asymptotic surface contains $3n(n-2)$ lines in the plane at infinity which are the intersections of the planes of inflexion of the cone $\phi_n = 0$, and contains, moreover, the curve of the n^{th} degree, in which the plane at infinity intersects the cone; hence, the number of points in which the asymptotic surface is met by an arbitrary line in the plane at infinity

$$= 3n(n-2) + n = n(3n-5).$$

For limitations of the number arising from the existence of singular points, see Painvin's second article.*

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* Crelle's *Journal*, vol. 65.

Method of Approximation.

. 475. Although it is necessary to know general methods of handling the equations of surfaces, yet in order to find the shape at particular points or at an infinite distance, it is most instructive for the student to employ peculiar methods to suit peculiar cases.

The method of approximation by transferring the origin to the particular point in question, and rejecting all terms which can be shewn to be small compared with those retained, gives immediately conical tangents or any other form which nearly coincides with a surface in the neighbourhood of a singular point.

The form of a surface at an infinite distance may be found by a careful consideration of the relative magnitude of the coordinates in the same manner as the author has treated the subject in his Treatise on Curve Tracing. The kind of consideration required may be seen by the following example.

476. *To find the plane and parabolic asymptotes of the surface whose equation is*

$$x^3 + y^3 + z^3 - 3xyz - 3a(yz + zx + xy) = 0.$$

The equation may be written

$$uv - a(u^2 - v) = 0,$$

where $u \equiv x + y + z$, $v \equiv x^2 + y^2 + z^2 - yz - zx - xy$.

If u^2 and v be of the same order of magnitude when x, y, z are very great, we have for a first approximation $u = 0$, and for a second $ua = 0$, the plane asymptote touching along a circle at infinity.

If u^2 be large compared with v, the first approximation gives $v = au$, and the next gives $v = a(u - a)$, which is a paraboloid of revolution.

The same results may be obtained by making the line $x = y = z$ one of the axes of coordinates, so that the equation becomes

$$\sqrt{(3)}\, x(y^2 + z^2) - a(2x^2 - y^2 - z^2) = 0,$$

in which, if x, y, z be of the same order of magnitude, $\sqrt{(3)}\,x+a=0$; and if x be large compared with y and z,

$$y^2+z^2-\frac{2a}{\sqrt{(3)}}\left\{x-\frac{a}{\sqrt{(3)}}\right\}=0.$$

The conical tangent at the origin is $y^2+z^2=2x^2$.

XVIII.

(1) Prove that the tangent plane to the surface $xyz=a^3$ forms with the coordinate planes a tetrahedron of constant volume.

(2) Find the equation of the tangent plane at any point of the surface $xyz+2abc=bcx+cay+abz$, and find the conical tangent at (a, b, c).

(3) If tangent planes be drawn at every point of the curve of intersection of the surface $a\,(yz+zx+xy)=xyz$, with a sphere whose centre is at the origin, shew that the sum of the three intercepts on the axes will be the same for all.

(4) A surface is given by the elimination of a between the equations $F(x, y, z, a)=0$ and $f(x, y, z, a)=0$; shew that the direction-cosines of the normal at a point (x, y, z) are in the ratio

$F'(x)f'(a)-f'(x)\,F'(a) : F'(y)\,f'(a)-f'(y)\,F'(a) : F'(z)f'(a)-f'(z)\,F'(a)$.

(5) The points on a conicoid, the normals at which intersect the normal at a fixed point, lie on a cone of the second degree, having its vertex at the fixed point.

(6) Prove that the projections on the plane of xy of the normals to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, at points whose distance from that plane is $c\cos a$, touch the curve $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}\sin^{\frac{2}{3}}a$.

(7) Given $(x^2+y^2+z^2+c^2-a^2)^2=4c^2(x^2+y^2)$, find the points the normals at which make angles a, β, γ with the axes, and the loci of points for which (i) γ is constant, (ii) a is equal to β.

(8) A chord of a conicoid is intersected by the normal at a given point of the surface, the product of the tangents of the angles subtended at the point by the two segments of the chords being invariable.

Prove that, O being the given point, and P, P' the intersections of the normal with two such chords in perpendicular planes containing the normal, the sum of the reciprocals of OP, OP' is invariable.

(9) Find the tangent cone at the origin to the surface

$$(x^2+y^2+ax)^2-(c^2-a^2)(x^2+z^2)=0;$$

and shew that as a diminishes and ultimately vanishes, the tangent cone

contracts, and ultimately becomes a straight line, and as a increases up to c, it expands, and finally becomes a plane.

(10) Shew that the 27 lines in a general cubic surface intersect in 135 points.

(11) Apply the method of Art. 448 to find the singular tangent planes of the wave surface.

(12) Shew that the normals to any scroll along a generating line lie on an hyperbolic paraboloid.

(13) If tangent planes at two points on a generating line of a scroll be at right angles, prove that the rectangle under the distances of the points of contact from the line of striction measured along that generating line will be constant.

(14) If a series of straight lines, generating a surface, be described according to a law such that the shortest distance between two consecutive lines is of a degree superior to the first, it will be at least of the third.

(15) Shew that the lines of striction of an hyperbolic paraboloid $\frac{y^2}{b} - \frac{z^2}{c} = x$ are its intersections with the planes $\frac{y^2}{b^3} - \frac{z^2}{c^3} = 0$.

(16) A straight line intersects at right angles the arc of a fixed circle, and turns about the tangent with half the angular velocity of the point of contact round the circle.

Prove that the surface so generated intersects itself on a straight line, and find the tangent planes at any point of this line.

Shew that the line of striction is a plane curve, whose plane is inclined to the plane of the circle at an angle $\tan^{-1} 2$.

(17) Find the asymptotic planes and the asymptotic surface of the conicoid $ax^2 + by^2 + cz^2 = 2x$.

(18) Shew that the coordinate planes are the three singular asymptotic planes of the surface $xyz = a^3$.

(19) From different points of the straight line $\frac{x}{a} = \frac{y}{b}$, $z = 0$, asymptotic straight lines are drawn to the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$; shew that they will all lie in the planes $\frac{x}{a} - \frac{y}{b} = \pm \frac{z}{c} \sqrt{2}$.

(20) Shew that the asymptotic planes to the surface

$$z(x^2 + y^2) - ax^2 - by^2 = 0,$$

are parallel to the plane xy, and that the locus of straight lines in these planes having contact of the second order at infinity is $z = a$, or $z = b$; and that the axis of z is an evanescent asymptotic cylinder.

(21) If the cone of asymptotic directions have a double side, shew that the surface will generally touch the plane at infinity, and that the section by this plane will have its inflexional tangents in the intersection with the tangent planes at the double side of the cone.

(22) Shew that the conicoid which determines the inflexional asymptotes of the surface, whose equation is $x^4 - y^2z^2 - 2a^2yz = 0$, is an hyperboloid of one or two sheets, the latter giving imaginary asymptotes.

(23) Discuss the form of the surface $z(x+y)^2 - a(x^2 - y^2) + b^2z = 0$ at an infinite distance.

(24) Shew that the asymptotic surface of $z(x+y)^2 - az^2 + bx^2 = 0$ is a parabolic cylinder.

(25) Shew that there is a conjugate line in the surface

$$a^2\{2(y^2+z^2) - x^2\}^2 = (y^2+z^2)(y^2+z^2-a^2)^2.$$

(25) Shew that the surface

$$(x^2 - z^2)(x^2 + 3y^2 - z^2 + 9a^2)^2 = \{6a(x^2+y^2-z^2) + 4a^3\}^2$$

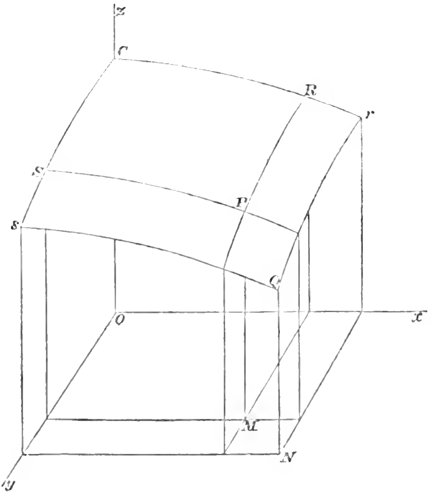
has a conjugate hyperbolic line in the plane of zx.

CHAPTER XIX.

VOLUMES, AREAS OF SURFACES, &c.

477. *To find the differential coefficients of the solid contained between a surface, given in rectangular coordinates, the coordinate planes, and planes parallel to them drawn through any point of the surface.*

Let x, y, z and $x + \Delta x$, $y + \Delta y$, $z + \Delta z$ be the coordinates of two points P and Q upon the surface.



Draw planes through P and Q parallel to the planes of yz, zx, and let V be the volume $CRPSOM$ cut off by these planes from the given solid. If ΔV be the increment of V,

when x is changed to $x+\Delta x$, while y remains constant, and a similar interpretation be given to the operation Δ_y, the volume $PrM=\Delta_x V$; also the volume $PQNM$, which is the increment of $\Delta_x V$ when y changes to $y+\Delta y$, $=\Delta_y(\Delta_x V)$, which is easily seen to be the same as $\Delta_x(\Delta_y V)$.

Let z_1, z_2 be the least and greatest values of z within the portion of the surface PQ, therefore $PQNM$ lies between $z_1\Delta x\Delta y$ and $z_2\Delta x\Delta y$;

$$\therefore\ \frac{\Delta_y\left(\dfrac{\Delta_x V}{\Delta x}\right)}{\Delta y} \text{ or } \frac{\Delta_x\left(\dfrac{\Delta_y V}{\Delta y}\right)}{\Delta x} \text{ lies between } z_1 \text{ and } z_2.$$

Proceeding to the limit, in which $z_1=z_2=z$, we obtain

$$\frac{d^2V}{dydx} \text{ or } \frac{d^2V}{dxdy}=z.$$

We may observe that, since the volume PrM is ultimately equal to the area $RM\times\Delta x$, the partial differential coefficient $\dfrac{dV}{dx}$ represents the area RM, and similarly $\dfrac{dV}{dy}$ the area SM.

478. The differential coefficient of the volume of a wedge of the solid contained between the planes of zx, xy, a plane through the axis of z, and a plane parallel to yOz may be obtained as follows.

Let V be the volume included between the planes zOx, xOy, the surface, the plane whose equation is $y=tx$, and a plane parallel to yOz through any point (x, y, z), then $\Delta_t V$ is the increment of V when t changes to $t+\Delta t$, x remaining constant, and is the volume which stands on a base whose area is $\frac{1}{2}x\Delta t.x$; $\Delta_x(\Delta_t V)$ is the increment of $\Delta_t V$ when x changes to $x+\Delta x$, and is the volume which stands on a base whose area is

$$\tfrac{1}{2}(x+\Delta x)^2\Delta t-\tfrac{1}{2}x^2\Delta t=(x+\tfrac{1}{2}\Delta x)\,\Delta x\,\Delta t;$$

hence, as before, $\dfrac{\Delta_x(\Delta_t V)}{\Delta x\,\Delta t}$ is between $z_1(x+\frac{1}{2}\Delta x)$ and $z_2(x+\frac{1}{2}\Delta x)$,

and, proceeding to the limit, $\dfrac{d^2V}{dx\,dt}=zx$.

479. *To find the differential coefficient of the portion of a surface given in rectangular coordinates, cut off by the coordinate planes, and planes parallel to them drawn through any point of the surface.*

Let P, Q be the points (x, y, z) and $(x+\Delta x, y+\Delta y, z+\Delta z)$, S the surface $PRCS$, cut off by the planes through P. $\Delta_x S$ is the surface Pr, which is the increment of S when x is changed to $x+\Delta x$.

$\Delta_y(\Delta_x S)$ is the surface PQ, which is the increment of $\Delta_x S$ when y is changed to $y+\Delta y$, and is evidently the same as $\Delta_x(\Delta_y S)$.

Let γ_1, γ_2 be the greatest and least inclinations of the tangent plane to the plane of xy for any point within the surface PQ.

Therefore PQ is intermediate between $\Delta x \Delta y \sec\gamma_1$ and $\Delta x \Delta y \sec\gamma_2$.

Hence $\dfrac{\Delta_y\left(\dfrac{\Delta_x S}{\Delta x}\right)}{\Delta y}$ or $\dfrac{\Delta_x\left(\dfrac{\Delta_y S}{\Delta y}\right)}{\Delta x}$ is intermediate between $\sec\gamma_1$ and $\sec\gamma_2$, which are, in the limit, each equal to $\sec\gamma$.

$$\text{Therefore, } \frac{d^2 S}{dy\,dx} \text{ or } \frac{d^2 S}{dx\,dy} = \sec\gamma = \sqrt{\left\{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2\right\}}.$$

480. If S be the surface contained between the plane zOx, and a plane whose equation is $y = tx$; we can shew, by proceeding as in Art 478, that

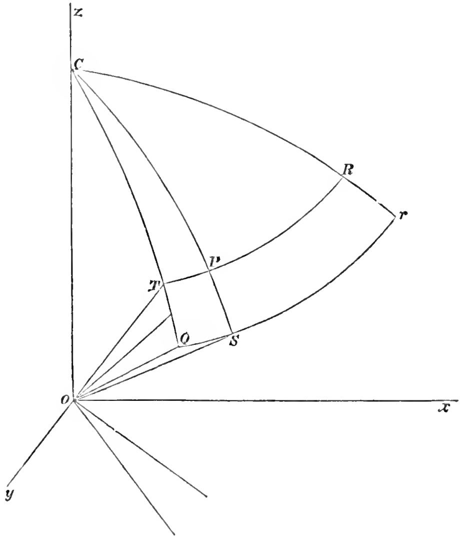
$$\frac{d^2 S}{dx\,dt} = \sqrt{\left[\left\{1+\left(\frac{dz}{dx}\right)^2\right\}x^2+\left(\frac{dz}{dt}\right)^2\right]}.$$

481. *To find the differential coefficients of the volume of a surface referred to polar coordinates.*

Let r, θ, ϕ be the polar coordinates of a point P in the surface, θ being measured from Oz, and ϕ from the plane zOx, and let V be the volume of the wedge of a cone contained between the planes zOx and zOP, and the given surface, the axis of the cone being Oz, and θ the semi-vertical angle.

$OPRrS$ is the increase of the volume when θ increases by $\Delta\theta$, ϕ remaining constant, therefore $OPRrS = \Delta_\theta V$.

$OPSQT$ is the increase of $\Delta_\theta V$ when ϕ becomes $\phi + \Delta\phi$, and therefore $= \Delta_\phi (\Delta_\theta V)$, and similarly $= \Delta_\theta (\Delta_\phi V)$.



If OP, OS, OQ, OT intersect a sphere, whose centre is O and radius OP, in P, s, q, t the volumes of $OPSQT$ and $OPsqt$ will be ultimately equal, and $Ps = r\Delta\theta$, $Pt = r \sin\theta . \Delta\phi$, therefore $\Delta_\phi (\Delta_\theta V)$ is ultimately equal to $\frac{1}{3} r^3 \sin\theta \, \Delta\phi \, \Delta\theta$;

$$\therefore \; \frac{d^2 V}{d\phi \, d\theta} = \tfrac{1}{3} r^3 \sin\theta.$$

482. *To find the differential coefficient of a surface referred to polar coordinates.*

Let r, θ, ϕ be the polar coordinates of P, and let S be the surface CPR,

$\Delta_\theta S$ is the increment Pr when θ changes to $\theta + \Delta\theta$,

$\Delta_\phi (\Delta_\theta S)$ is the increment PQ when ϕ changes to $\phi + \Delta\phi$.

Let ψ_1, ψ_2 be the least and greatest inclinations of tangent planes at points taken on the surface $PSQT$, to tangent planes at the corresponding points of the surface $Psqt$ in the construction of the last article, then the ratio of the surfaces $PSQT$ and $Psqt$ lies between $1 : \cos\psi_1$, and $1 : \cos\psi_2$, each of which becomes ultimately $r : p$, where p is the perpendicular from O on the tangent at P;

$$\therefore\ \Delta_\phi\,(\Delta_\theta S) = \frac{r^3}{p}\sin\theta\Delta\phi\Delta\theta \text{ ultimately};$$

$$\therefore\ \frac{d^2S}{d\phi\,d\theta} = \frac{r^3}{p}\sin\theta,$$

$$\text{and } \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^4\sin^2\theta}\left(\frac{dr}{d\phi}\right)^2 \text{ (Art. 464)};$$

$$\therefore\ \frac{d^2S}{d\phi\,d\theta} = r\left[\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}\sin^2\theta + \left(\frac{dr}{d\phi}\right)^2\right]^{\frac{1}{2}}.$$

483. *To find the volume of a closed surface, the boundaries of which are portions of known surfaces, given by equations in Cartesian coordinates.*

Let (x, y, z) be a point P within the closed surface, and let Δx, Δy, Δz be the lengths of the edges of a small parallelepiped, whose faces are parallel to the coordinate planes, the volume of this elementary parallelepiped will be $\Delta x\Delta y\Delta z$, if the axes be rectangular.

We imagine the volume to be made up of an infinite number of such elements, each of which is supposed indefinitely small, and in order to obtain the volume we have to sum these elements, and we must be guided by the form of the surfaces in our choice of the order in which we propose to effect the summation. We can give general directions only, leaving to the student's ingenuity the task of adapting them to particular cases.

If we commence by summing the elements, for which x, y have constant values, we shall obtain the parellelepiped $(z_2 - z_1)\,\Delta x\Delta y$, since the incomplete elements near the boundaries of the surface vanish, compared with the parallelepiped upon $\Delta x\Delta y$, when Δx and Δy are indefinitely diminished; $z_2 - z_1$ can

be expressed in terms of x and y by means of the equations of the bounding surfaces. This supposes the closed surface to be pierced by the ordinate through $(x, y, 0)$ in only two points; if it were pierced $2n$ times, the first summation would give $\Sigma_1^n (z_{2r} - z_{2r-1})\, dxdy$; we shall not further consider such cases.

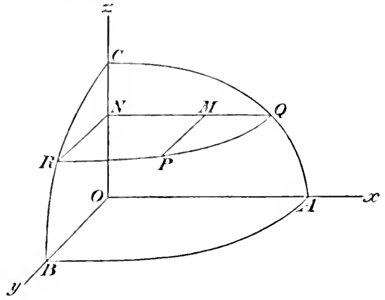
If we next sum the parallelepipeds for all values of y, keeping x constant, we shall obtain the sum of all the elements which lie between two planes at distances x and $x + \Delta x$ from the plane of yz. The first and last of the parallelepipeds must vanish; therefore the summation must generally be made between values of y obtained from the equation $z_2 - z_1 = 0$, x being constant; let y_1, y_2 be those values of y, supposing only two to exist; the whole sum will then be obtained by summing these sheets of elements between values of x obtained from the equation $y_2 - y_1 = 0$.

In the case of a closed surface, which is pierced by no straight line in more than two points, the process is expressed thus:

$$\begin{aligned}
\text{volume} &= \int_{x_1}^{x_2} dx \left\{\int_{y_1}^{y_2} dy \left(\int_{z_1}^{z_2} dz\right)\right\} \\
&= \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} \{f_2(x, y) - f_1(x, y)\}\, dy \\
&= \int_{x_1}^{x_2} dx \{\phi(x, y_2) - \phi(x, y_1)\} \\
&= \psi(x_2) - \psi(x_1).
\end{aligned}$$

484. The student will have to determine in every particular case the best order in which to make the summation of the elements; in some cases it will be advisable to take elementary slices of the surface, instead of the elementary parallelepipeds, as when the area of a plane section is known.

Thus, in the case of an ellipsoid, the area of a section RPQ is $\pi QN, RN$, and a slice of the thickness $dz = \frac{\pi ab}{c^2}(c^2 - z^2)\, dz$, whence the volume is $\frac{\pi ab}{c^2}\int_{-c}^{+c}(c^2 - z^2)\, dz = \frac{4}{3}\pi abc$.



485. He must also judge whether it is advisable to use other coordinates than those in which the equation of the surface is given.

Thus, the equation of an anchor-ring being

$$(x^2+y^2+z^2+c^2-a^2)^2-4c^2(x^2+y^2)=0,$$

if we make $x^2+y^2=r^2$, $z^2=a^2-(r-c)^2$, we can sum the elements which have their projections on the circular ring $2\pi r dr$, and the volume is

$$\int_{c-a}^{c+a} 4\pi r dr \sqrt{\{a^2-(r-c)^2\}} = \int_{-a}^{+a} 4\pi (r'+c)\, dr' \sqrt{(a^2-r'^2)} = 2\pi c.\pi a^2.$$

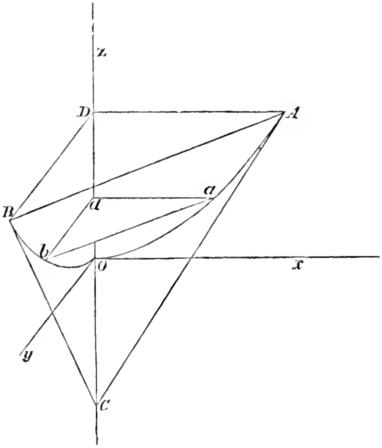
486. *To find the volume contained between the surface whose equation is $(x+y)^2=4az$, the tangent plane at a given point, and the planes of zx and yz.*

Let the given point be (f, g, h), the equation of the tangent plane is $x+y=\sqrt{\left(\frac{a}{h}\right)}(z+h)$; the volume required is $\iiint dx\,dy\,dz$, the limits being from $z=\sqrt{\left(\frac{h}{a}\right)}(x+y)-h$ to $\frac{(x+y)^2}{4a}$, then from $y=0$ to $y=2\sqrt{(ah)}-x$, since the tangent plane meets the surface where $(x+y)^2-4\sqrt{(ah)}(x+y)+4ah=0$, lastly from $x=0$ to $x=2\sqrt{(ah)}$. The volume is

$$\iint \frac{1}{4a}\{x+y-2\sqrt{(ah)}\}^2\, dy\, dx$$

$$=\int -\frac{1}{12a}\{x-2\sqrt{(ah)}\}^3\, dx = \frac{\{2\sqrt{(ah)}\}^4}{4\,.\,12a} = \tfrac{1}{3}ah^2.$$

This result may be verified thus. Let AOB be the surface, ACB the tangent plane along the line AB, ADB a plane parallel to xOy, adb any section of the surface parallel to xOy.



$$\text{Then area } adb : \text{area } ADB :: ad^2 : AD^2 :: Od : OD;$$

$$\text{therefore volume } AOBD = \int_0^h 2ah \,.\, \frac{z}{h}\, dz = ah^2;$$

$$\text{also volume } ACBD = \tfrac{1}{3}2ah\,.\,2h = \tfrac{4}{3}ah^2;$$

$$\text{hence the volume required is } \frac{ah^2}{3}.$$

487. *To find the volume of the elliptic paraboloid* $\dfrac{y^2}{b} + \dfrac{z^2}{c} = 2x$, *cut off by the plane* $lx + my + nz = p$.

Perform the integration in the order x, y, z,

$$x_1 = \frac{y^2}{2b} + \frac{z^2}{2c}, \quad x_2 = \frac{p - my - nz}{l}.$$

For a given value of z, the values of y at the curve of intersection are given by the equation $x_1 = x_2$,

$$\text{or } y^2 + \frac{2bm}{l} y + \frac{b}{c} z^2 - \frac{2b}{l}(p - nz) = 0, \qquad (1)$$

of which y_1, y_2 are the roots, and z must be taken between the limits which correspond to $y_1 = y_2$, that is z_1, z_2 are the roots of the equation

$$\frac{b}{c} z^2 - \frac{2b}{l} (p - nz) = \frac{b^2 m^2}{l^2}. \qquad (2)$$

$$\text{The volume} = \int_{z_1}^{z_2} \int_{y_1}^{y_2} \left(\frac{p - my - nz}{l} - \frac{y^2}{2b} - \frac{z^2}{2c}\right) dy\, dz$$

$$= \frac{1}{2b} \int_{z_1}^{z_2} \int_{y_1}^{y_2} (y - y_1)(y_2 - y)\, dy\, dz \quad \text{by (1)},$$

$$= \frac{1}{2b} \int_{z_1}^{z_2} \int_{y_1}^{y_2} \{(y - y_1)(y_2 - y_1) - (y - y_1)^2\}\, dy\, dz$$

$$= \frac{1}{2b} \int_{z_1}^{z_2} \tfrac{1}{6} (y_2 - y_1)^3\, dz,$$

but $(y_2 - y_1)^2 = (y_2 + y_1)^2 - 4y_2 y_1 = \dfrac{4b}{c} (z - z_1)(z_2 - z)$ by (2), therefore the volume

$$= \frac{1}{12b} \int_{z_1}^{z_2} 8 \left(\frac{b}{c}\right)^{\frac{3}{2}} \{(z_2 - z_1)(z - z_1) - (z - z_1)^2\}^{\frac{3}{2}}\, dz$$

$$= \tfrac{2}{3} \frac{b^{\frac{1}{2}}}{c^{\frac{3}{2}}} \int_{-\gamma}^{\gamma} (\gamma^2 - u^2)^{\frac{3}{2}}\, du, \text{ where } 2\gamma = z_2 - z_1,\ u = z - \tfrac{1}{2}(z_1 + z_2)$$

$$= \tfrac{4}{3} \frac{b^{\frac{1}{2}}}{c^{\frac{3}{2}}} \gamma^4 \int_0^{\frac{1}{2}\pi} \cos^4\theta\, d\theta, \text{ putting } u = \gamma \sin\theta,$$

$$= \frac{\pi}{4} \cdot \frac{b^{\frac{1}{2}}}{c^{\frac{3}{2}}} \frac{(z_2 - z_1)^4}{2^4},$$

and
$$\tfrac{1}{4}(z_2 - z_1)^2 = \frac{c^2 n^2}{l^2} + \frac{2cp}{l} + \frac{bcm^2}{l^2};$$

$$\therefore \text{volume} = \frac{\pi}{4} \sqrt{(bc)}\, \frac{(2pl + bm^2 + cn^2)^2}{l^4}.$$

The student may verify this result by the summation of elementary slices bounded by planes parallel to the given plane.

488. *To find the volume contained between surfaces given by polar coordinates.*

The volume of an elementary parallelepiped is

$$r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

If we integrate this expression from $r=r_1$ to $r=r_2$, r_1, r_2 being the radii of the bounding surfaces corresponding to θ, ϕ, we obtain a frustum of a pyramid, the angular breadths of whose faces are $d\theta$, $d\phi$, intercepted between the two surfaces or the two sheets of the same surface; its volume is $\frac{1}{3}\sin\theta d\theta d\phi\,(r_2^{\,3}-r_1^{\,3})$, the radii being given in terms of θ and ϕ.

If now we integrate, considering ϕ and $\phi+d\phi$ constant, from $\theta=\theta_1$ to $\theta=\theta_2$, θ_1, θ_2 being given in terms of ϕ by the boundaries of the volume considered, we obtain the portion included between the planes inclined to zOx at angles ϕ and $\phi+d\phi$,

$$=d\phi\int_{\theta_1}^{\theta_2}\tfrac{1}{3}\,(r_2^{\,3}-r_1^{\,3})\,\sin\theta\,d\theta.$$

The whole volume is found by integrating from $\phi=\phi_1$ to $\phi=\phi_2$, the extreme planes between which the volume is included.

The volume is therefore $\displaystyle\int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2}\tfrac{1}{3}\,(r_2^{\,3}-r_1^{\,3})\,\sin\theta\,d\theta\,d\phi.$

489. *To find the volume of a sphere cut off by three planes through the centre.*

Let the radius of the sphere be a, O its centre, and let ABC be the spherical triangle cut off. Take OC for the axis of z, and a plane perpendicular to AOB for that of zx.

The equation of the plane AOB will be $\cos\phi=\tan\alpha\cot\theta$, and the limits of integration will be

$$\begin{aligned} r&=0 \quad\text{to}\quad r=a,\\ \theta&=0 \quad\text{to}\quad \theta=\cot^{-1}(\cot\alpha\cos\phi),\\ \phi&=-\beta \quad\text{to}\quad \phi=C-\beta;\end{aligned}$$

$$\therefore \text{ the volume } = \frac{a^3}{3}\iint\sin\theta\,d\theta\,d\phi$$

$$=\frac{a^3}{3}\int_{-\beta}^{C-\beta}\left\{1-\frac{\cos\alpha\cos\phi}{\sqrt{(1-\cos^2\alpha\sin^2\phi)}}\right\}d\phi$$

$$=\frac{a^3}{3}\left[C-\sin^{-1}\{\cos\alpha\sin(C-\beta)\}-\sin^{-1}(\cos\alpha\sin\beta)\right]$$

$$=\frac{a^3}{3}\,(A+B+C-\pi),$$

since $\cos B=\cos\alpha\sin(C-\beta)$ and $\cos A=\cos\alpha\sin\beta$.

We have given this as an example of the determination of the limits in the case of polar coordinates, but the result is obtained immediately from the area of the spherical triangle, the volume required being the sum of an infinite number of pyramids whose vertices are in the centre, the volume of any one of which is $\frac{1}{3}a\,dS$, and the whole volume $= \frac{1}{3}a \times$ area of the spherical triangle.

490. *To find the volume of a wedge of a sphere cut off by a right circular cylinder, a diameter of whose base is a radius of the sphere.*

Let the equation of the sphere be $\rho^2 + z^2 = a^2$, and that of the cylinder $\rho = a \cos\phi$.

$$\text{The volume is } \int_0^\alpha \int_0^{a\cos\phi} 2\rho\,\sqrt{(a^2-\rho^2)}\,d\rho\,d\phi$$

$$= \int_0^\alpha \tfrac{2}{3}\,(a^3 - a^3\sin^3\phi)\,d\phi$$

$$= \tfrac{2}{3}a^3\left\{\alpha - \tfrac{1}{4}\int_0^\alpha (3\sin\phi - \sin 3\phi)\,d\phi\right\}$$

$$= \tfrac{2}{3}a^3\left\{\alpha - \tfrac{3}{4}(1-\cos\alpha) + \tfrac{1}{12}(1-\cos 3\alpha)\right\}.$$

$$\text{The surface } = \iint \sqrt{\left\{\rho^2 + \left(\frac{dz}{d\phi}\right)^2 + \rho^2\left(\frac{dz}{d\rho}\right)^2\right\}}\,d\rho\,d\phi,$$

between the same limits,

$$= \iint \rho\,\sqrt{\left(1 + \frac{\rho^2}{a^2-\rho^2}\right)}\,d\rho\,d\phi = \iint \frac{a\rho\,d\rho}{\sqrt{(a^2-\rho^2)}}\,d\phi$$

$$= a^2\int_0^\alpha (1-\sin\phi)\,d\phi = a^2(\alpha - 1 + \cos\alpha).$$

491. The following method of dividing a surface into elements was employed by Gauss in treating of the curvature of surfaces.

The coordinates of a point are considered as functions of two parameters α, β, the elimination of which would lead to the equation of the surface.

If α vary while β is constant the corresponding points on the surface will lie on a curve, and a system of curves will will be formed by giving β successive constant values.

Another system of curves will be obtained by making β vary while α is constant.

The element dS is the quadrilateral figure whose sides are portions of the curves which correspond to constant values α, $\alpha + d\alpha$ in one system and β, $\beta + d\beta$ in the other.

Let l, m, n be the direction cosines of the normal to the surface at a point in the element dS; ldS is the projection of the element on the plane of yz, let this be $PQRS$, the coordinates of the angular points in order being y, z;

$$y + \frac{dy}{d\alpha}d\alpha,\ \ z + \frac{dz}{d\alpha}d\alpha;$$

$$y + \frac{dy}{d\alpha}d\alpha + \frac{dy}{d\beta}d\beta,\ \ z + \frac{dz}{d\alpha}d\alpha + \frac{dz}{d\beta}d\beta;$$

$$\text{and } y + \frac{dy}{d\beta}d\beta,\ \ z + \frac{dz}{d\beta}d\beta.$$

The equation of PQ is $\dfrac{\eta - y}{\dfrac{dy}{d\alpha}} = \dfrac{\zeta - z}{\dfrac{dz}{d\alpha}}$, and it is easily seen that PQ and SR are ultimately parallel; the perpendicular from S on PQ is

$$\frac{\left(\dfrac{dy}{d\beta}\dfrac{dz}{d\alpha} - \dfrac{dz}{d\beta}\dfrac{dy}{d\alpha}\right)d\beta}{\left\{\left(\dfrac{dy}{d\alpha}\right)^2 + \left(\dfrac{dz}{d\alpha}\right)^2\right\}^{\frac{1}{2}}},$$

$$\text{also } PQ = \left\{\left(\frac{dy}{d\alpha}\right)^2 + \left(\frac{dz}{d\alpha}\right)^2\right\}^{\frac{1}{2}} d\alpha;$$

$$\therefore\ ldS = \left(\frac{dy}{d\beta}\frac{dz}{d\alpha} - \frac{dz}{d\beta}\frac{dy}{d\alpha}\right)d\alpha d\beta,$$

and if we write $Ad\alpha d\beta$, $Bd\alpha d\beta$, $Cd\alpha d\beta$ for the projections on the three coordinate planes,

$$dS = (A^2 + B^2 + C^2)^{\frac{1}{2}}\, d\alpha d\beta.$$

The surface-integral $\iint(lu + mv + nw)dS$, where u, v, w are given functions of the position of dS may thus be expressed in terms of the parameters α, β, viz.

$$\iint(Au + Bv + Cw)\, d\alpha d\beta.$$

492. *To find the surface of an ellipsoid expressed in elliptic coordinates.*

Let the equation of the ellipsoid be

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\lambda^2 - \beta^2} + \frac{z^2}{\lambda^2 - \gamma^2} = 1,$$

and let μ, ν be primary semi-axes of the hyperboloids, which are the elliptic coordinates.

By Art. 283,

$$x^2 = \frac{\lambda^2\mu^2\nu^2}{\beta^2\gamma^2}, \quad y^2 = -\frac{(\lambda^2 - \beta^2)(\mu^2 - \beta^2)(\nu^2 - \beta^2)}{(\gamma^2 - \beta^2)\beta^2}, \quad z^2 = \ldots;$$

$$\therefore\ y\frac{dy}{d\mu} = -\frac{\mu(\lambda^2 - \beta^2)(\nu^2 - \beta^2)}{(\gamma^2 - \beta^2)\beta^2}, \quad z\frac{dz}{d\nu} = \frac{\nu(\lambda^2 - \gamma^2)(\mu^2 - \gamma^2)}{(\gamma^2 - \beta^2)\gamma^2};$$

$$\therefore\ yz\left(\frac{dy}{d\nu}\frac{dz}{d\mu} - \frac{dz}{d\nu}\frac{dy}{d\mu}\right) = \frac{\mu\nu(\lambda^2 - \beta^2)(\lambda^2 - \gamma^2)(\mu^2 - \nu^2)}{\beta^2\gamma^2(\gamma^2 - \beta^2)},$$

$$dS = \frac{1}{l}\left(\frac{dy}{d\nu}\frac{dz}{d\mu} - \frac{dz}{d\nu}\frac{dy}{d\mu}\right)d\mu\,d\nu$$

$$= \frac{\lambda^2}{pxyz} \times \frac{\mu\nu(\lambda^2 - \beta^2)(\lambda^2 - \gamma^2)(\mu^2 - \nu^2)}{\beta^2\gamma^2(\gamma^2 - \beta^2)}\,d\mu\,d\nu$$

$$= \frac{\lambda(\mu^2 - \nu^2)\sqrt{\{(\lambda^2 - \beta^2)(\lambda^2 - \gamma^2)\}}}{p\sqrt{\{(\mu^2 - \beta^2)(\gamma^2 - \mu^2)(\beta^2 - \nu^2)(\gamma^2 - \nu^2)\}}}\,d\mu d\nu$$

$$= \frac{(\mu^2 - \nu^2)\sqrt{\{(\lambda^2 - \mu^2)(\lambda^2 - \nu^2)\}}}{\sqrt{\{(\mu^2 - \beta^2)(\gamma^2 - \mu^2)(\beta^2 - \nu^2)(\gamma^2 - \nu^2)\}}}\,d\mu d\nu, \text{ Art. 286.}$$

The area of the surface cut off by four confocal hyperboloids, for which $\mu = \mu_1$ and μ_2, $\nu = \nu_1$ and ν_2 is

$$\int_{\mu_1}^{\mu_2} \mu^2 M d\mu \times \int_{\nu_1}^{\nu_2} N d\nu - \int_{\mu_1}^{\mu_2} M d\mu \times \int_{\nu_1}^{\nu_2} \nu^2 N d\nu,$$

where $M = \sqrt{\dfrac{\lambda^2 - \mu^2}{(\mu^2 - \beta^2)(\gamma^2 - \mu^2)}}$, $N = \sqrt{\dfrac{\lambda^2 - \nu^2}{(\beta^2 - \nu^2)(\gamma^2 - \nu^2)}}$.

493. If the position of a point be given as the intersection of three surfaces

$$F(x, y, z) = \alpha, \quad G(x, y, z) = \beta, \quad \text{and } H(x, y, z) = \gamma,$$

the expression for a volume may be obtained similarly as follows; when γ is constant, the variation of α and β determines

a surface of which an elementary portion is $(A^2+B^2+C^2)^{\frac{1}{2}}\,d\alpha d\beta$, and the equation of the tangent plane at this element is

$$A(\xi-x)+B(\eta-y)+C(\zeta-z)=0,$$

the perpendicular on which plane from a point determined by α, β, $\gamma+d\gamma$ is

$$\frac{\left(A\dfrac{dx}{d\gamma}+B\dfrac{dy}{d\gamma}+C\dfrac{dz}{d\gamma}\right)d\gamma}{(A^2+B^2+C^2)^{\frac{1}{2}}};$$

hence the volume of the elementary parallelepiped, whose opposite faces correspond to γ, $\gamma+d\gamma$ constant, &c., is

$$\left(A\frac{dx}{d\gamma}+B\frac{dy}{d\gamma}+C\frac{dz}{d\gamma}\right)d\alpha\,d\beta\,d\gamma,$$

and the volume $=\iiint J d\alpha\,d\beta\,d\gamma=\iiint\frac{1}{J'}\,d\alpha\,d\beta\,d\gamma$,

$$\text{where } J=\begin{vmatrix}\dfrac{dx}{d\alpha}, & \dfrac{dy}{d\alpha}, & \dfrac{dz}{d\alpha}\\[2ex] \dfrac{dx}{d\beta}, & \dfrac{dy}{d\beta}, & \dfrac{dz}{d\beta}\\[2ex] \dfrac{dx}{d\gamma}, & \dfrac{dy}{d\gamma}, & \dfrac{dz}{d\gamma}\end{vmatrix} \text{ and } J'=\begin{vmatrix}\dfrac{d\alpha}{dx}, & \dfrac{d\alpha}{dy}, & \dfrac{d\alpha}{dz}\\[2ex] \dfrac{d\beta}{dx}, & \dfrac{d\beta}{dy}, & \dfrac{d\beta}{dz}\\[2ex] \dfrac{d\gamma}{dx}, & \dfrac{d\gamma}{dy}, & \dfrac{d\gamma}{dz}\end{vmatrix}.$$

494. *To find the volume of a solid whose bounding surfaces are given by tetrahedral coordinates.*

Let ξ, η, ζ be coordinates referred to rectangular axes of a point whose tetrahedral coordinates are x, y, z, w.

Since x, y, z are linear functions of ξ, η, ζ,

$$\iiint d\xi\,d\eta\,d\zeta=C\iiint dx\,dy\,dz,$$

and if V be the volume of the tetrahedron of reference

$$\iiint d\xi\,d\eta\,d\zeta=V;$$

but the limits for the tetrahedron are, since $x+y+z+w=1$,

$$z=0 \text{ to } w=0 \text{ or } z=1-x-y,$$

$$y=0 \text{ to } y=1-x,$$

$$x=0 \text{ to } x=1,$$

and with these limits $\iiint dx\,dy\,dz=\frac{1}{6}$; therefore $6V=C$.

Hence, if $F(x, y, z, w) = 0$ be the equation of any closed surface, the volume will be $6V\iiint dxdydz$, the limits of the integration being obtained from

$$F(x, y, z, 1 - x - y - z) = 0.$$

This method is due to Slesser.*

XIX.

(1) Find the volume of the surface $xy + yz + zx - a^2 = 0$, cut off by the plane $x + y + z = c$.

(2) State limits which can be used to find the volume of a closed conicoid whose equation is $ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 1$.

(3) Find the portion of the cylinder $x^2 + y^2 - 2rx = 0$, intercepted between the planes $ax + by + cz = 0$ and $a'x + by + cz = 0$.

(4) State between what limits the summation of $dxdydz$ must be taken in order to obtain the volume of the cone whose equation is $x^2 + y^2 = (a - z)^2$, cut off by the planes $x = 0$ and $x = z$.

(5) Find the volume contained between the surfaces

$$y^2 + z^2 = 4ax, \text{ and } x - z = a.$$

(6) Prove that the volume included between the surfaces $r = a$, $z = 0$, $\theta = 0$, $z = mr\cos\theta$ is $\frac{2}{3}ma^3$, r and θ being polar coordinates in the plane xy.

(7) Shew that the volume enclosed by the surfaces $x^2 + y^2 = az$, $x^2 + y^2 = ax$, and $z = 0$ is $\dfrac{3\pi a^3}{16}$, and draw a figure representing the progress of summation.

(8) Prove that the volume included between a cylinder $y^2 = 2rx - x^2$, a paraboloid $\dfrac{x^2}{a} + \dfrac{y^2}{b} = 2z$ and the plane of xy is $\pi r^4\left(\dfrac{5}{8a} + \dfrac{3}{8b}\right)$.

(9) Prove that the volume cut off from the cone

$$ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0,$$

by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ is $\frac{4}{3}\pi abc\,(1 - k)$, the curves of intersection of the cone and ellipsoid being ellipses, and k given by the equation

$$\frac{gh}{gh - uf} + \frac{hf}{hf - vg} + \frac{fg}{fg - wh} = \frac{1}{k^2}.$$

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* *Quart. Jour. of Math.*, vol. II.

(10) Prove that the volume cut off by the plane $y = k$ from the surface $a^2x^2 + b^2z^2 = 2(ax + bz)\,y^2$ is $\dfrac{\pi k^5}{10ab}$.

(11) A cavity is just large enough to allow of the complete revolution of a circular disk of radius c, whose centre describes a circle of the same radius c, while the plane of the disk is constantly parallel to a fixed plane, and perpendicular to that of the circle in which the centre moves. Shew that the volume of the cavity is $\dfrac{2c^3}{3}(3\pi + 8)$.

(12) Two cones have a common vertex in the centre of an ellipsoid, and their bases are curves in which the surface is intersected by planes parallel to the same principal plane, prove that the volume of the ellipsoid contained between the cones varies as the distance between the planes.

(13) Prove that the volume contained between the plane $z = (c - x)\cot\alpha$ and the surface $xz^2 + (x - c)(x^2 + y^2) = 0$ is

$$\frac{\pi c^3}{48}(3\cot\alpha\operatorname{cosec}\alpha - 2\cos^3\alpha - 3\log\cot\tfrac{1}{2}\alpha).$$

(14) The volume contained between the surface

$$\frac{z^4}{c^4} - 2\frac{z^2}{c^2}\left(\frac{x}{a} + \frac{y}{b}\right) + \frac{z^2}{c^2}\left(\frac{x}{a} - \frac{y}{b}\right)^2 + \frac{4xy}{ab} = 0,$$

and either of the planes yz or xz is $\dfrac{\pi abc}{32}$.

(15) Shew that the whole volume of the surface whose equation is $(x^2 + y^2 + z^2)^2 = cxyz$ is equal to $\dfrac{c^3}{360}$.

(16) Investigate the form of the surface whose equation is

$$\{(x^2 + z^2)^{\frac{1}{2}} - a\}^2 + y^2 = \frac{a^2}{4\pi^2}\left(\tan^{-1}\frac{z}{x}\right)^2,$$

and shew that its volume between values of $\tan^{-1}\dfrac{z}{x}$ from 0 to 2π is $\frac{2}{3}\pi^3a^3$.

(17) Shew that the volume of the closed portion of the surface whose equation is $4a(y^2 + z^2 - 4a^2) + (x^2 - a^2)(3z + 10a) = 0$ is $\frac{27}{81}\cdot\frac{1}{4}\pi(5a)^3$.

(18) If ΔS be an element of the surface of an ellipsoid at any point, and A the area of a section by a plane drawn through the centre, parallel to the tangent plane at that point, prove that the limit of $\Sigma\dfrac{\Delta S}{A} = 4$, the summation being taken over the whole surface.

Find ΔS in terms of α, β, if $x = a\cos\alpha$, $y = b\sin\alpha\cos\beta$, and $z = c\sin\alpha\sin\beta$.

(19) If S be a closed surface, dS an element about P, at a distance r from a fixed point O, ϕ the angle which the normal drawn inwards makes

with OP, shew that the volume contained by the surface $= \frac{1}{3}\iint r \cos\phi dS$, the summation being extended over the whole surface.

O being the centre of an ellipsoid, apply the formula to find its volume, interpreting geometrically the steps of the integration.

(20) Shew that $\iint \frac{x^2 dS}{p}$ extended over the surface of an ellipsoid is equal to $\frac{1}{5}\left(3 + \frac{a^2}{b^2} + \frac{a^2}{c^2}\right) \times$ volume of the ellipsoid.

(21) Prove that the area of a closed surface, no plane section of which has singular points, may be expressed by the definite integral

$$\int_{-\pi}^{\pi} \int_{0}^{\pi} \frac{r^3 \sin\phi \, d\phi d\theta}{p},$$

where p is the perpendicular from the origin upon the tangent plane.

(22) If each element of a closed surface be multiplied by $\frac{\mu}{r^2}\cos\phi$, where r is the distance of the element from a point O, and ϕ is the angle between the direction of r and the normal to the surface measured outwards, shew that the sum of all such products is 0 or $4\pi\mu$, according as O is without or within the surface.

(23) If r be the distance from a point O of any element dS of a spherical surface, determine the form of the function $f(r)$ when $\iint \frac{f(r)}{r} dS$, the summation being effected over the whole surface of the sphere, is constant for all positions of O within the sphere.

(24) Shew that the shortest distances between generating lines of the same system drawn at the extremities of diameters of the principal elliptic section of the hyperboloid, whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, lie on the surfaces whose equations are $\frac{cxy}{x^2+y^2} = \pm \frac{abz}{a^2-b^2}$. Prove also that the volume included between these surfaces and the hyperboloid is

$$\frac{abc}{3}\left(\frac{a^4-b^4}{a^2b^2} + 8\log\frac{a}{b}\right).$$

CHAPTER XX.

TORTUOUS CURVES. CURVATURE. TORTUOSITY.

495. We have already shewn that curves may be considered as the complete or partial intersection of surfaces, but in the investigation of the equations of tangents, osculating planes &c. we shall also look upon a curve as the locus of points which satisfy more general laws, the algebraical statement of which assumes the form of equations between the coordinates of any point of the curve and variable parameters, the number of equations being two more than the number of parameters.

Instances of the latter mode of representation of a curve occur in dynamical problems, in which the curve is defined by equations between the coordinates of the position of a particle and the time of its arrival at that position.

If the parameters were eliminated from the equations connecting the coordinates and parameters, the result would be two final equations which would be the equations of two surfaces whose complete or partial intersections would be the curve in question.

496. If the coordinates of any point on a curve can be expressed as functions of a single parameter t, so that for each value of t there is a single value of each coordinate, the curve is called *unicursal*.

497. As an example of an unicursal curve, we may take the Helix, which is generated by the uniform motion of a point along a generating line of a right cylinder as the generating line revolves with uniform angular velocity about the axis of the cylinder.

If we take the axis for the axis of z, and the axis of x through the generating point at any initial time, θ the angle through which the generating line has revolved when the point has moved through a space z on the generating line, we have, for the co-ordinates of the point, a being the radius of the cylinder,

$$x = a\cos\theta,\ \ y = a\sin\theta,\ \ z = na\theta;$$

here θ is the variable parameter, and the curve is the intersection of the surfaces $x^2 + y^2 = a^2$, and $y = x\tan\dfrac{z}{na}$.

498. In order to explain the terms employed in the examination of curves which are not plane, we shall consider such curves as the limits of polygons whose sides are indefinitely small; and we observe that the plane which contains any two consecutive sides of the polygon of which the curve is the limit, does not generally contain the next side.

The term double curvature, as is remarked by Thomson and Tait,* is not a proper expression, since there are not two curvatures; and the property, that the plane in which the curvature is taking place at any point changes as the point changes, would be better represented by calling the curve *tortuous* and the measure of the corresponding property *tortuosity*.

499. *Osculating plane.* The plane containing two sides of the polygon of which a tortuous curve is the limit is in its ultimate position an *osculating plane* of the curve.

500. *Normal Plane.* Any side of the polygon in its limiting position is a tangent to the curve, and a plane drawn perpendicular to the tangent though the point of contact is a *normal plane*, being the locus of all the normals at the point.

501. *Principal Normal.* The particular normal which lies in the osculating plane is called the *principal normal*.

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* *Natural Philosophy*, Art. 7.

502. *Binormal.** The normal which is perpendicular to the osculating plane is called a *binormal*, being perpendicular to two elements of the curve.

503. *Polar Developable.*† Let an equilateral polygon be inscribed in a curve, of which consecutive sides are PQ, QR, RS, ST, and let p, q, r, s be the middle points of these sides.

Let Aap, Bbq, Ccr be planes perpendicular to these sides, forming the polygon $ABCD$ by their intersections.

If the sides PQ, QR, ... be diminished indefinitely, their directions are ultimately those of tangents to the curve, the planes Aap, Bbq, ... are ultimately normal planes to the curve, the planes PQR, QRS, ... are osculating planes, and the surface generated by the plane elements Aab, Bbc, Ccd, ... is ultimately the developable surface enveloped by the normal planes of the curve, of which $ABCD$ ... is ultimately the edge of regression.

The developable enveloped by the normal planes is called the *Polar Developable.*

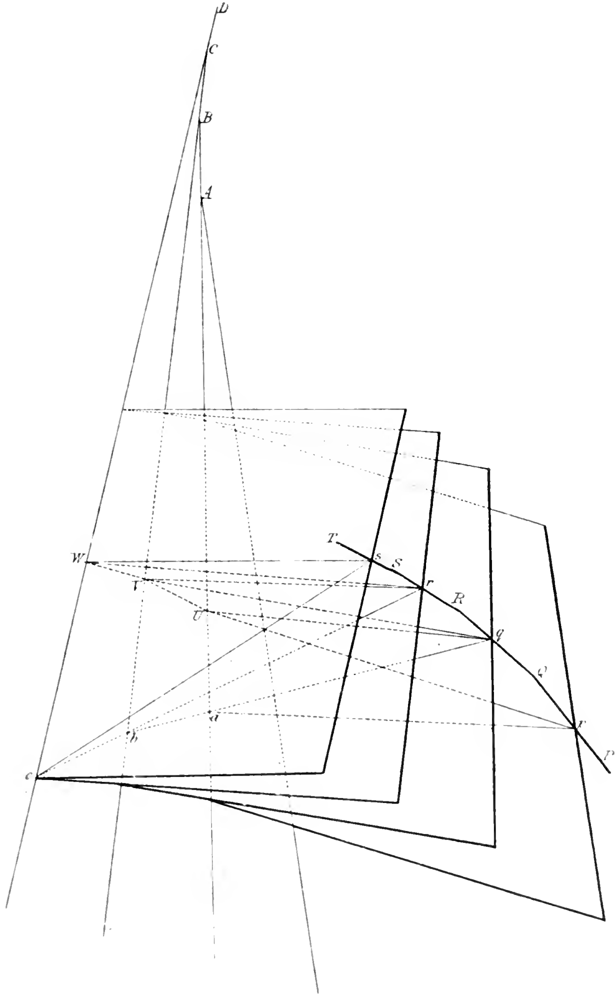
504. *Circle of Curvature.* A circle can be described containing the points P, Q, R; when the sides are indefinitely diminished, this circle lies in the osculating plane, and its curvature may be taken as the measure of curvature of the curve in the osculating plane. Let the plane PQR meet Aa in U, and let pU, qU be joined, then since PQ is perpendicular to the plane Apa, it is perpendicular to pU, similarly QR is perpendicular to qU, U is therefore the centre of the circle through PQR.

Therefore the centre of the circle of curvature is the point of intersection of two consecutive normal planes and the osculating plane.

505. *Polar Line.* Draw pa, qa to any point in Aa, then, since $Pp = Qp$, a is equally distant from P and Q, and similarly from Q and R, and therefore from every point in the circle of

* St. Venant. † Monge.

To face p 342.



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curvature. The line of intersection of two consecutive normal planes is called by Monge the *polar line.*

506. *Angle of Contingence.* The angle pUq, which is equal to the angle between the two consecutive sides PQ, QR of the polygon, is ultimately equal to the angle between two consecutive tangents, and is called the *angle of contingence.*

507. *Sphere of Curvature.* Any point in Aa is equally distant from P, Q and R; also any point in Bb is equally distant from Q, R, and S; therefore their point of intersection is equally distant from the four points P, Q, R, S.

Hence, it follows that a sphere can be described whose centre is B, and which contains the four points P, Q, R, S, this sphere is ultimately the sphere which has the closest possible contact with the curve, since no sphere can be made to pass through more than four arbitrary points, it is therefore called the *sphere of curvature*: the locus of its centre is the edge of regression of the polar developable.

508. *Evolutes.* It has been shewn, Art. 443, that, if a be *any* point in the intersection of the planes normal to PQ, QR, at their middle points p, q, ap and aq will be equal and will make equal angles with Aa. Produce qa to meet Bb in b; then a string, placed in the position bap, would remain in that position if subject to tension, since the tensions of the portions ab, ap resolved parallel to Aa would be equal, and, if its extremity were then moved from p to q it would occupy the position baq. Similarly, if rb be produced to c in Cc, and if sc be produced to d in Dd.

If we proceed to the limit, it follows that a string may be stretched upon the polar developable in such a manner that the free end, starting from any point in the curve, would describe the curve, if the string were unwrapped from the surface so that the part in contact with the surface remained stationary. The portion in contact lies on a curve called the *evolute.*

Also, since the position of the line pa is arbitrary, the curve which is the limit of a, b, c, d,... will change its position ac-

cording to the position of a, hence the number of evolutes is infinite.

All the evolutes of a curve are geodesic lines of the polar developable.

509. *Angle of Torsion.* The plane pUq perpendicular to AUa contains the sides PQ, QR, and the plane qVr perpendicular to BVb contains the sides QR, RS, and, since qU, qV are perpendicular to the line of intersection QR of the two planes, the angle UqV is their angle of inclination.

This angle, which is ultimately the angle between consecutive osculating planes, is called the *angle of torsion.*

Also, since a circle goes round $BVUq$, the angles UqV and UBV are equal, and the angle of torsion of the curve PQR,...... is equal to the angle of contingence of the edge of regression of the polar developable.

510. *Locus of Centres of Circular Curvature not an Evolute.* Since qU will not, if produced, pass through V, because qU and qV include an angle in the same normal plane, the locus of the centres of circular curvature is not one of the evolutes.

511. *Rectifying Developable.* If through every point of a curve a plane be drawn perpendicular to the corresponding principal normal, these planes will envelope a torse on which the curve will be a geodesic line, since its osculating plane will contain the normal to the surface at every point; if therefore the torse be developed into a plane, the curve will be developed into a straight line. On account of this property the torse is called the *Rectifying Developable.*

512. *Rectifying Line.* The line of intersection of two consecutive planes, enveloping the rectifying developable, is called the *rectifying line* for any point of the curve, being the line about which the curve must turn at that point in order to become straight, when the torse is developed into a plane.

It may be observed that the rectifying line is not generally coincident with the binormal, which is the normal perpendicular to the osculating plane.

In the figure at p. 342 the surface whose edge of regression is the limit of ABC... is the rectifying surface to the curve which is the limit of abc.... Aa is the rectifying line at a, and the binormal does not coincide with the rectifying line unless pa be perpendicular to Aa, or a be the centre of curvature of the involute of abc...

513. If the polygon $PQRS$... were transformed into a plane polygon by turning the portion $QRST$... through the angle of torsion VqU about QR, and the portion RST... about RS through the corresponding angle of torsion, the inclination of any side ST in the new position in the plane of PQR would be inclined to PQ at an angle equal to the sum of the inclinations of the sides taken in order, and estimated in the same direction.

Proceeding to the limit, we see that if, as a point moves along a tortuous curve, at every position which the point assumes the curve be turned about the tangent line through the angle of torsion, the curve will be replaced by a plane curve, such that the inclination of the tangents at the starting point and any other point will be the sum of all the angles of contingence; if, therefore, ε be taken for the angle between the tangents in the plane curve, $d\varepsilon$ will be the angle of contingence corresponding to the extremity of the arc traversed by the moving point.

514. *Rate of Torsion.* The rate per unit of length of arc at which the osculating plane twists about the tangent line at any point, called the *rate of torsion*, is measured by the limit of the ratio of the angle of torsion to the arc at the extremities of which the osculating planes are taken.

If, as we pass from PQ to QR, see figure, p. 342, QR be turned in the plane PQR so that PQR is a straight line, and the plane QRS be then turned through the angle VqU, the process being repeated along the whole of a given arc, the perimeter will become rectified, and the inclination of the last to the first position of the plane containing two elements will be the sum of all angles such as VqU between the extremities of the arc so rectified.

Proceeding to the limit, it follows that, if osculating planes be taken along the curve, and the elements of the arc be rectified in each osculating plane in order, the angle between the first and final positions of the osculating plane when the curve is so rectified will be the sum of the angles of torsion throughout the arc.

If, therefore, τ be this angle, $d\tau$ will be the angle of torsion, corresponding to the point at which the last osculating plane is drawn.

515. *Integral and Average Curvature.** The integral curvature of any portion of a curve is the angle through which the tangent will have turned as we pass from one extremity to the other, the average curvature is its whole curvature divided by its length.

Let a sphere of unit radius have its centre at a fixed point, and let radii be drawn parallel to the tangents to the curve at successive points, the length of the curve traced on the sphere by the extremities of the radii measures the integral curvature of the portion of the curve considered, and the average curvature is the integral curve divided by the length of the curve.

516. *Integral and Average Tortuosity.* These are respectively the angle through which the osculating plane has turned in passing from end to end of any portion of a curve, and this angle divided by the length of the arc considered.

On the sphere described in the last article let a curve be described by the poles of the tangents to the curve which measures the integral curvature, the length of this curve measures the integral tortuosity, and this length divided by the length of the arc of the tortuous curve the average tortuosity.

Tangents.

517. *Tangent to a curve at a given point.*

Let s, $s + \Delta s$ be the lengths measured along the arc of a curve from a given point to the points P and Q, whose coordinates are x, y, z and $x + \Delta x$, $y + \Delta y$, $z + \Delta z$, and let $c =$ chord PQ.

* Thomson and Tait, *Nat. Phil.*, Arts. 10–12.

As Q approaches to and ultimately coincides with P, the chord PQ and arc Δs become equal, PQ is the direction of the tangent at P, and the direction cosines of PQ, viz. $\frac{\Delta x}{c}, \frac{\Delta y}{c}, \frac{\Delta z}{c}$ become ultimately $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$.

The equations of the tangent are therefore

$$\frac{\xi - x}{dx} = \frac{\eta - y}{dy} = \frac{\zeta - z}{dz}.$$

Also since $c^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$,

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2.$$

(1) Let the equations of the curve be given in terms of a variable parameter θ, in the form

$$x = \phi(\theta), \quad y = \psi(\theta), \quad z = \chi(\theta),$$

then $dx : dy : dz = \phi'(\theta) : \psi'(\theta) : \chi'(\theta)$,

and the equations of the tangent at a point corresponding to θ are

$$\frac{\xi - x}{\phi'(\theta)} = \frac{\eta - y}{\psi'(\theta)} = \frac{\zeta - z}{\chi'(\theta)}.$$

(2) Let the equations be those of surfaces containing the curve $F(\xi, \eta, \zeta) = 0$, and $G(\xi, \eta, \zeta) = 0$.

Then, at any point P of the curve,

$$F'(x)\,dx + F'(y)\,dy + F'(z)\,dz = 0,$$

and $G'(x)\,dx + G'(y)\,dy + G'(z)\,dz = 0$;

whence the equations of the tangent PQ may be written

$$F'(x)(\xi - x) + F'(y)(\eta - y) + F'(z)(\zeta - z) = 0,$$

and $G'(x)(\xi - x) + G'(y)(\eta - y) + G'(z)(\zeta - z) = 0$,

which equations represent analytically the fact that the tangent to the curve lies in the tangent plane to each surface at the common point P.

(3) If the surfaces, the intersection of which gives the curve, be cylindrical surfaces whose sides are parallel to the two axes of z and y, and their equations be $\eta = f(\xi)$, $\zeta = \phi(\xi)$, the equations of the tangent will be

$$\eta - y = f'(x)(\xi - x),$$

$$\zeta - z = \phi'(x)(\xi - x).$$

These equations are the analytical representation of the fact that the projections of the tangent to the curve on the coordinate planes of xy, zx are the tangents to the respective projections of the curve; which is obviously true, since the projections of P and Q have their ultimate coincidence simultaneously with that of P and Q.

518. *To find the directions of the branches of the curve of intersection of two surfaces at a multiple point of the curve.*

The equations of the surfaces being

$$F(\xi, \eta, \zeta) = 0, \text{ and } G(\xi, \eta, \zeta) = 0,$$

and (x, y, z) being a multiple point P on the curve, let

$$\frac{\xi - x}{\lambda} = \frac{\eta - y}{\mu} = \frac{\zeta - z}{\nu} = r \tag{1}$$

be the equations of a line through P; the points in which this line meets the surfaces are given by the equations

$$\left.\begin{aligned} F(x + \lambda r,\ y + \mu r,\ z + \nu r) = 0 \\ \text{and } G(x + \lambda r,\ y + \mu r,\ z + \nu r) = 0 \end{aligned}\right\}, \tag{2}$$

there are an infinite number of directions which give two values of r equal to zero, since the curve has a multiple point at P; therefore the two equations

$$\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy} + \nu \frac{d}{dz}\right) F(x, y, z) = 0, \tag{3}$$

$$\text{and } \left(\lambda \frac{d}{dx} + \mu \frac{d}{dy} + \nu \frac{d}{dz}\right) G(x, y, z) = 0, \tag{4}$$

must be one or both identically satisfied, or else they must not be independent equations.

i. If only one of the equations (3) and (4) be identically satisfied, suppose this to be (3); then (x, y, z) will be a multiple point on the surface $F(\xi, \eta, \zeta) = 0$; and, if this be a double point, the line (1) must be one of the tangents whose directions are given by

$$\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy} + \nu \frac{d}{dz}\right)^2 F(x, y, z) = 0;$$

and, since it lies in the tangent plane to $G(\xi, \eta, \zeta) = 0$,

$$\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy} + \nu \frac{d}{dz}\right) G(x, y, z) = 0.$$

These equations give the directions of the two tangent lines, which are the intersections of the conical tangent to the first surface with the tangent plane to the second; and, similarly, for higher degrees of multiplicity.

ii. If (3) and (4) be both identically satisfied, the line (1) will be in any of the directions of common tangents to $F(\xi, \eta, \zeta) = 0$ and $G(\xi, \eta, \zeta) = 0$; the directions are therefore given by

$$\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy} + \nu \frac{d}{dz}\right)^s F(x, y, z) = 0$$

$$\text{and} \quad \left(\lambda \frac{d}{dx} + \mu \frac{d}{dy} + \nu \frac{d}{dz}\right)^t G(x, y, z) = 0,$$

where s and t are the degrees of multiplicity of the multiple points of the two surfaces at (x, y, z).

iii. If neither (3) nor (4) be identically satisfied, but the two equations be identical so as to be satisfied by an infinite number of values of $\lambda : \mu : \nu$, there will be a surface $AF + BG = 0$, which will pass through the intersection of $F = 0$ and $G = 0$, for which $\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy} + \nu \frac{d}{dz}\right)(AF + BG) = 0$ will be identically satisfied, if $-\frac{B}{A}$ be the value of the equal ratios $\frac{F'(x)}{G'(x)}$, $\frac{F'(y)}{G'(y)}$, and $\frac{F'(z)}{G'(z)}$.

In this case, therefore, $\lambda : \mu : \nu$ is determined by one of the equations (3), (4), and

$$\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy} + \nu \frac{d}{dz}\right)^2 (AF + BG)(x, y, z) = 0.$$

If in any of these cases two values of $\lambda : \mu : \nu$ be equal, there will be either a point of osculation or cusp on the curve.

519. As an example of case iii. in the last Article, suppose we wish to find the directions of the tangents at the point

$(a, 0, 0)$ in the curve of intersection of the hyperboloid and hyperbolic paraboloid, whose equations are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

$$\text{and } \frac{y^2}{l} - \frac{z^2}{l'} = 2(x - a).$$

At this point the surfaces have a common tangent plane, whose equation is $x = a$; the third surface, on which $(a, 0, 0)$ is a multiple point, is in this case the cone

$$\left(\frac{x}{a} - 1\right)^2 + \left(\frac{1}{b^2} + \frac{1}{al}\right) y^2 - \left(\frac{1}{c^2} + \frac{1}{al'}\right) z^2 = 0,$$

and the direction cosines of the tangents to the curve are given by

$$\frac{\lambda^2}{a^2} + \left(\frac{1}{b^2} + \frac{1}{al}\right)\mu^2 - \left(\frac{1}{c^2} + \frac{1}{al'}\right)\nu^2 = 0 \text{ and } \lambda = 0.$$

520. *Normal plane of a curve at a given point.*

The normal plane being perpendicular to the tangent to the curve, its equation is

$$(\xi - x)\,dx + (\eta - y)\,dy + (\zeta - z)\,dz = 0.$$

521. *To find the edge of regression of the polar developable of a curve.*

The edge of regression is the locus of the intersection of three consecutive normal planes to the curve.

The equation of the normal plane at (x, y, z) is

$$(\xi - x)\,dx + (\eta - y)\,dy + (\zeta - z)\,dz = 0, \qquad (1)$$

that of the normal plane at a consecutive point is found by writing in this equation $x + dx$ for x, &c., the line of intersection of the two normal planes will lie in the plane

$$(\xi - x)\,d^2x + (\eta - y)\,d^2y + (\zeta - z)\,d^2z - (dx)^2 - (dy)^2 - (dz)^2 = 0. (2)$$

Again, writing $x + dx$ for x, &c., we obtain a plane in which the line of intersection of the second and third normal planes lies,

$$(\xi - x)\,d^3x + (\eta - y)\,d^3y + (\zeta - z)\,d^3z$$
$$- 3\,(dx\,d^2x + dy\,d^2y + dz\,d^2z) = 0, \qquad (3)$$

and the coordinates of the point of the edge of regression satisfy these three equations. If we eliminate x, y, z from the equations (1), (2), (3) and the equations of the curve, we shall obtain the two equations of the edge of regression.

The line of which (1) and (2) are the equations is Monge's polar line, which is the axis of the osculating circle.

The point given by the three equations (1), (2), (3) is the centre of spherical curvature corresponding to the point (x, y, z) of the curve.

522. *To find the differential coefficient of the arc referred to polar coordinates.*

Transforming to polar coordinates

$$x = r \sin\theta \cos\phi = \rho \cos\phi,$$

$$y = r \sin\theta \sin\phi = \rho \sin\phi,$$

$$z = r \cos\theta,$$

$$\rho = r \sin\theta,$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{d\rho}{d\theta}\right)^2 + \rho^2 \left(\frac{d\phi}{d\theta}\right)^2,$$

$$\left(\frac{dz}{d\theta}\right)^2 + \left(\frac{d\rho}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2;$$

$$\therefore \left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2 + r^2 \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2.$$

The equation is easily obtained geometrically by observing that ultimately

$$(\Delta s)^2 = (\Delta r)^2 + (r\Delta\theta)^2 + (r \sin\theta\Delta\phi)^2.$$

Also, if p be the perpendicular from the pole upon the tangent, and ψ the angle between r and the tangent, $p = r \sin\psi$,

$$\text{and } \frac{\Delta s}{\Delta r} = \sec\psi \text{ ultimately, } \therefore \left(\frac{ds}{dr}\right)^2 = \frac{r^2}{r^2 - p^2}.$$

Osculating Plane.

523. *Equation of the osculating plane.*

The osculating plane may be considered as the plane which passes through three consecutive points, whose coordinates are x, y, z; $x + dx$, ... and $x + 2dx + d^2x$... .

Let the equation of the osculating plane be

$$A(\xi - x) + B(\eta - y) + C(\zeta - z) = 0;$$

$$\therefore\ A dx + B dy + C dz = 0,$$

$$\text{and } A(2dx + d^2x) + B(2dy + d^2y) + C(2dz + d^2z) = 0,$$

$$\text{or } Ad^2x + Bd^2y + Cd^2z = 0,$$

hence the equation is

$$\begin{vmatrix} \xi - x, & \eta - y, & \zeta - z \\ dx, & dy, & dz \\ d^2x, & d^2y, & d^2z \end{vmatrix} = 0.$$

It may be noted that the equations of the tangent and osculating plane are of the same form, whether the axes be rectangular or oblique.

524. It should be observed with respect to the notation used above that if x, y, z be supposed given as functions of t, and we take points corresponding to values t, $t+\tau$, $t+2\tau$, which is the same as making t the independent variable, the values of x for $t+\tau$ and $t+2\tau$ are

$$x + \frac{dx}{dt}\tau + \frac{d^2x}{dt^2}\frac{\tau^2}{2} + \ldots$$

$$\text{and } x + \frac{dx}{dt}2\tau + \frac{d^2x}{dt^2}\frac{(2\tau)^2}{2} + \ldots;$$

and if the first be written $x + \Delta x$, the second will be

$$x + \Delta x + \Delta(x + \Delta x) \text{ or } x + 2\Delta x + \Delta^2 x;$$

hence if d be written for Δ, where τ is indefinitely diminished, $dx = \dfrac{dx}{dt}\tau$ and $d^2x = \dfrac{d^2x}{dt^2}\tau^2$ ultimately.

525. As an exercise the student should find the equation of the osculating plane, considered as given by any of the following definitions:

i. As a plane containing a tangent and a point indefinitely near the point of contact.

ii. As a plane containing a tangent and parallel to a consecutive tangent.

iii. As a plane which has a closer contact with the curve than any other plane.

In employing the definition ii. he may shew that the shortest distance between the tangents at the extremity of any arc ds is generally of the order of ds^3.

526. *Direction cosines of the binormal.*

The direction cosines of the binormal, which is perpendicular to the osculating plane, are in the ratio

$$dyd^2z - dzd^2y : dzd^2x - dxd^2z : dxd^2y - dyd^2x,$$

and the sum of the squares of these expressions

$$=\{(dx)^2+(dy)^2+(dz)^2\}\{(d^2x)^2+(d^2y)^2+(d^2z)^2\}-(dxd^2x+dyd^2y+dzd^2z)^2$$
$$=(ds)^2\{(d^2x)^2+(d^2y)^2+(d^2z)^2\}-(dsd^2s)^2,$$
$$\text{since } (ds)^2=(dx)^2+(dy)^2+(dz)^2,$$

hence the direction cosines of the binormal are

$$\pm\frac{dyd^2z-dzd^2y}{ds\{(d^2x)^2+(d^2y)^2+(d^2z)^2-(d^2s)^2\}^{\frac{1}{2}}}, \text{ \&c.}$$

If the pivot of the hands of a watch, with its face in the osculating plane, be supposed placed at the centre of curvature of the curve so that the extremities of the hands move in the direction in which ds is measured, the + sign must be used, when the direction of the pivot is chosen as the positive direction of the binormal. In fact, if the pivot make an acute angle with the axis of x, it is evident that, when s is measured in the direction of the motion of the hands, $\frac{dz}{dy}$ will increase with s, and $dy\,d^2z - dz\,d^2y$ will be positive

527. *To find the condition that an osculating plane may be stationary, and that a curve whose equations are given may be plane.*

The condition that an osculating plane may be stationary, or that the osculating planes at two consecutive points on the curve may coincide, will be satisfied for any point (x, y, z), if the osculating plane at that point contain a fourth point, for which the value of x is $x + 3dx + 3d^2x + d^3x$.

If we write this value and the corresponding values of y

and z for ξ, η, and ζ in the equation of the osculating plane, we shall obtain the equation

$$(dy\,d^2z - dz\,d^2y)\,d^3x + (dz\,d^2x - dx\,d^2z)\,d^3y + (dx\,d^2y - dy\,d^2x)\,d^3z = 0$$

as the condition required.

If this condition hold at every point of the curve, the curve will be plane, and the equation of its plane will be that of the osculating plane.

528. When the curve is given by means of the equations of two surfaces of which it is the intersection, the calculations required for the determination of the osculating plane may be conveniently conducted as follows:

529. *To find the osculating plane of the curve of intersection of two surfaces, whose equations are given.*

Let $\phi(x, y, z) = 0$, $\phi'(x, y, z) = 0$ be the given equations, then, using the notation of Art. 420,

$$Udx + Vdy + Wdz = 0,$$

$$U'dx + V'dy + W'dz = 0.$$

Let D, E, F denote the determinants $\left\| \begin{matrix} U, & V, & W \\ U', & V', & W' \end{matrix} \right\|$;

$$\therefore \frac{dx}{D} = \frac{dy}{E} = \frac{dz}{F} = k \text{ suppose,}$$

$$\text{whence } d^2x = kdD + Ddk,$$

$$d^2y = kdE + Edk,$$

$$d^2z = kdF + Fdk;$$

$$\therefore dyd^2z - dzd^2y = k^2(EdF - FdE),$$

hence the equation of the osculating plane is

$$(EdF - FdE)(\xi - x) + \ldots = 0.$$

530. *Equation of the osculating plane in terms of the equations of the tangent planes to the surfaces.*

Employing the notation of the preceding article, we see that

$$DU + EV + FW = 0;$$

$$\therefore UdD + VdE + WdF + DdU + EdV + FdW = 0,$$

$$\text{and}\quad dU = dx\,\frac{dU}{dx} + dy\,\frac{dU}{dy} + dz\,\frac{dU}{dz}$$

$$= k\left(D\,\frac{d}{dx} + E\,\frac{d}{dy} + F\,\frac{d}{dz}\right)\frac{d\phi}{dx}$$

$$= k\,\frac{d}{dx}\,\Gamma(\phi),$$

if Γ denote the operation in the brackets, in the performance of which D, E, F are considered constant;

$$\therefore\ DdU + EdV + FdW = k\Gamma^2(\phi),$$

$$\text{hence}\quad UdD + VdE + WdF = -k\Gamma^2(\phi),$$

$$\text{similarly}\quad U'dD + V'dE + W'dF = -k\Gamma^2(\phi');$$

$$\therefore\ EDF - FdE = k\{U\Gamma^2(\phi') - U'\Gamma^2(\phi)\},$$

and the equation of the osculating plane becomes

$$\Gamma^2(\phi')\{U(\xi - x) + V(\eta - y) + W(\zeta - z)\} = \Gamma^2(\phi)\{U'(\xi - x) + \ldots\}.$$

531. *To find the osculating plane of the intersection of two concentric and coaxial conicoids.*

Let the equations be

$$\begin{aligned} ax^2 + by^2 + cz^2 &= 1, \\ \alpha x^2 + \beta y^2 + \gamma z^2 &= 1, \end{aligned} \qquad (1)$$

$$D = 4(b\gamma - c\beta)\,yz = Ayz,\quad E = Bzx,\quad F = Cxy,$$

$$EdF - FdE = E^2 d\left(\frac{F}{E}\right) = BCz^2x^2 d\left(\frac{y}{z}\right)$$

$$= BCx^2(zdy - ydz) = kBCx^2(Ez - Fy),$$

$$\text{and by (1)}\quad Ez - Fy = 4(\alpha - a)\,x;$$

$$\therefore\ EdF - FdE = 4kBC(\alpha - a)\,x^3,$$

and the equation of the osculating plane is

$$\frac{\alpha - a}{A}\,x^3(\xi - x) + \frac{\beta - b}{B}\,y^3(\eta - y) + \frac{\gamma - c}{C}\,z^3(\zeta - z) = 0,$$

which may be reduced to

$$\frac{BC}{(\beta - b)(\gamma - c)}\,x^3\xi + \frac{CA}{(\gamma - c)(\alpha - a)}\,y^3\eta + \frac{AB}{(\alpha - a)(\beta - b)}\,z^3\zeta + 1 = 0,$$

532. Or, by the method of Art. 530, since

$$\tfrac{1}{2}\Gamma^2(\phi) = D^2a + E^2b + F^2c,$$

the equation may be written

$$(D^2\alpha + E^2\beta + F^2\gamma)(ax\xi + by\eta + cz\zeta - 1)$$
$$-(D^2a + E^2b + F^2c)(\alpha x\xi + \beta y\eta + \gamma z\zeta - 1) = 0,$$

and the coefficient of

$$\xi = \tfrac{1}{4}(E^2C - F^2B)\,x = \tfrac{1}{4}BC(Ez - Fy)\,x^2$$
$$= BC(\alpha - a)\,x^3, \text{ as before.}$$

533. *To find the condition for a stationary osculating plane of the curve of intersection of two surfaces.*

The equation of an osculating plane is

$$(EdF - FdE)(\xi - x) + \ldots = 0,$$

the line of intersection of this plane with the next consecutive osculating plane is in the plane

$$(Ed^2F - Fd^2E)(\xi - x) + \ldots - (EdF - FdE)\,dx - \ldots = 0\,;$$

the last three terms are identically zero, since $dx = kD$, and in order that the two osculating planes should coincide,

$$\frac{Ed^2F - Fd^2E}{EdF - FdE} = \frac{Fd^2D - Dd^2F}{FdD - DdF} = \frac{Dd^2E - Ed^2D}{DdE - EdD},$$

which are clearly equivalent to one distinct equation; and each of the fractions is equal to $\dfrac{d^2D\,(Ed^2F - Fd^2E) + \ldots}{d^2D\,(EdF - FdE) + \ldots}$, the numerator of which vanishes,

$$\therefore\; d^2D\,(EdF - FdE) + \ldots = 0.$$

Principal Normal.

534. *To find the equations of the principal normal at any point of a curve.*

The principal normal is perpendicular to the tangent line and also the binormal, the direction cosines of which are proportional to dx, dy, dz, and $dy\,d^2z - dz\,d^2y$, $dz\,d^2x - dx\,d^2z$, $dx\,d^2y - dy\,d^2x$ respectively.

Now we have identically

$$d^2x\,(dy\,d^2z - dz\,d^2y) + d^2y\,(dz\,d^2x - dx\,d^2z) + d^2z\,(dx\,d^2y - dy\,d^2x = 0\,;$$

and if we make s the independent variable,

$$d^2x\,dx + d^2y\,dy + d^2z\,dz = d^2s\,ds = 0.$$

These two equations shew that direction cosines of the principal normal are proportional to $\frac{d^2x}{ds^2}, \frac{d^2y}{ds^2}, \frac{d^2z}{ds^2}$, and with a general independent variable, its equations are

$$\frac{\xi - x}{\frac{d}{d\theta}\left(\frac{dx}{ds}\right)} = \frac{\eta - y}{\frac{d}{d\theta}\left(\frac{dy}{ds}\right)} = \frac{\zeta - z}{\frac{d}{d\theta}\left(\frac{dz}{ds}\right)}.$$

535. *If from any point in a curve equal distances be measured along the curve and its tangent, the limiting position of the line joining the extremities of these distances is the principal normal.*

From the point (x, y, z) let equal distances σ be measured along the curve and the tangent to the points Q, T. The coordinates of Q are $x + \frac{dx}{ds}\sigma + \left(\frac{d^2x}{ds^2} + \varepsilon\right)\frac{\sigma^2}{2}$, &c. and those of T $x + \frac{dx}{ds}\sigma$, &c., ε vanishing in the limit.

The equations of the line QT are

$$\frac{\xi - x - \frac{dx}{ds}\sigma}{\frac{d^2x}{ds^2} + \varepsilon} = \frac{\eta - y - \frac{dy}{ds}\sigma}{\frac{d^2y}{ds^2} + \varepsilon'} = \frac{\zeta - z - \frac{dz}{ds}\sigma}{\frac{d^2z}{ds^2} + \epsilon''};$$

therefore the limiting position of QT is the principal normal.

Cauchy proposed, as a definition of the Principal Normal at any point, the limiting position of the line joining the points on the curve and tangent, whose distances from the point of contact measured along the curve and tangent respectively are equal, by which means the definition was made independent of the osculating and normal planes.

Measure of Curvature.

536. *To find the radius of curvature at any point of a tortuous curve.*

The reciprocal of the radius of curvature is the measure of curvature, or the rate per unit of length at which the tangent

to the curve changes its direction. If ρ be the radius of curvature at a point P, and $d\epsilon$ be the angle of contingence corresponding to the arc ds, $\frac{1}{\rho} = \frac{d\epsilon}{ds}$.

Draw Op, Oq of unit length through the origin parallel to the tangents at P and Q the extremities of the arc ds, join pq; then, since the plane pOq is parallel to the osculating plane, pq, which is ultimately perpendicular to Op, is parallel to the principal normal.

The cosines of the angle made by Op and Oq with the axis of x are $\frac{dx}{ds}$ and $\frac{dx}{ds} + d\frac{dx}{ds}$, and, if l, m, n be the direction cosines of pq, projecting $OpqO$ on the axis of x, we have

$$\frac{dx}{ds} + d\epsilon . l - \left(\frac{dx}{ds} + d\frac{dx}{ds}\right) = 0,$$

since $pq = d\epsilon$ ultimately.

$$\therefore \; l = \rho\frac{d^2x}{ds^2}, \text{ and, similarly, } m = \rho\frac{d^2y}{ds^2}, \; n = \rho\frac{d^2z}{ds^2};$$

$$\therefore \; \frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2.$$

If C be the centre of curvature at P, the projection of OPC on $Ox = x + \rho l$, hence the coordinates of the centre of curvature are

$$x + \rho^2\frac{d^2x}{ds^2}, \; y + \rho^2\frac{d^2y}{ds^2}, \text{ and } z + \rho^2\frac{d^2z}{ds^2}.$$

537. The radius of curvature may also be found without projections, as follows:

Let λ, μ, ν and $\lambda + \Delta\lambda$, $\mu + \Delta\mu$, $\nu + \Delta\nu$ be the direction cosines of the tangents at the points P and Q, whose coordinates are x, y, z and $x + \Delta x$, $y + \Delta y$, $z + \Delta z$; and let $\Delta\epsilon$ be the angle between the tangents

$$\cos\Delta\epsilon = \lambda(\lambda + \Delta\lambda) + \mu(\mu + \Delta\mu) + \nu(\nu + \Delta\nu),$$

$$\text{and } (\lambda + \Delta\lambda)^2 + (\mu + \Delta\mu)^2 + (\nu + \Delta\nu)^2 - \lambda^2 - \mu^2 - \nu^2 = 0;$$

$$\therefore \; 2(\lambda\Delta\lambda + \mu\Delta\mu + \nu\Delta\nu) + (\Delta\lambda)^2 + (\Delta\mu)^2 + (\Delta\nu)^2 = 0;$$

$$\therefore\ 2(1-\cos\Delta\varepsilon) = (\Delta\lambda)^2 + (\Delta\mu)^2 + (\Delta\nu)^2;$$

$\therefore$ ultimately, when PQ is indefinitely diminished,

$$(d\varepsilon)^2 = (d\lambda)^2 + (d\mu)^2 + (d\nu)^2;$$

$$\therefore\ \frac{(ds)^2}{\rho^2} = \left(d\frac{dx}{ds}\right)^2 + \left(d\frac{dy}{ds}\right)^2 + \left(d\frac{dz}{ds}\right)^2.$$

If s be not the independent variable,

$$d\frac{dx}{ds} = \frac{ds\,d^2x - dx\,d^2s}{(ds)^2}, \text{ \&c.};$$

$$\therefore\ \text{since } (dx)^2 + (dy)^2 + (dz)^2 = (ds)^2,$$

$$\text{and } dx\,d^2x + dy\,d^2y + dz\,d^2z = ds\,d^2s;$$

$$\frac{(ds)^4}{\rho^2} = (d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2.$$

538. The student should, as an exercise, find the radius and centre of curvature, when the latter is considered as the point of intersection of two consecutive normal planes and the osculating plane.

Measure of Tortuosity.

539. *To find the measure of tortuosity of a tortuous curve.*

Let l, m, n and $l+dl$, $m+dm$, $n+dn$ be the direction cosines of the binormals at two points P, Q, whose distance along the curve is ds. Draw unit lengths Op, Oq parallel to the two binormals, and let λ, μ, ν be the direction cosines of pq; the angle $qOp = d\tau$ is the angle between the osculating planes, and $\frac{d\tau}{ds}$, the rate at which the osculating plane turns round the tangent line per unit of arc, is the measure required, which we shall call $\frac{1}{\sigma}$.

Projecting $OpqO$ on the axis of x,

$$l + d\tau.\lambda - (l+dl) = 0;$$

$$\therefore\ d\tau.\lambda = dl \text{ and } \lambda = \sigma\frac{dl}{ds};$$

$$\therefore\ \frac{1}{\sigma^2} = \left(\frac{dl}{ds}\right)^2 + \left(\frac{dm}{ds}\right)^2 + \left(\frac{dn}{ds}\right)^2,$$

which may also be obtained as in Art. 537. σ is sometimes called the radius of torsion at P, but it is better to look upon $\frac{1}{\sigma}$ as the measure of tortuosity.

540. The measure of tortuosity may be expressed in another form.

Since $l : m : n :: X : Y : Z$, where $X = dy\,d^2z - dz\,d^2y$, and similar expressions for Y, Z,

$$ldx + mdy + ndz = 0,$$

$$ld^2x + md^2y + nd^2z = 0\,;$$

$$\therefore\ dl\,dx + dm\,dy + dn\,dz = 0,$$

$$\text{and } dl.l + dm.m + dn.n = 0\,;$$

$$\therefore\ \frac{dl}{mdz - ndy} = \frac{dm}{ndx - ldz} = \frac{dn}{ldy - mdx}$$

$$= \frac{dl\,d^2x + dm\,d^2y + dn\,d^2z}{lX + mY + nZ}$$

$$= -\frac{ld^3x + md^3y + nd^3z}{lX + mY + nZ} = -\frac{Xd^3x + Yd^3y + Zd^3z}{X^2 + Y^2 + Z^2},$$

and $(mdz - ndy)^2 + \ldots$

$$= (l^2 + m^2 + n^2)\,\{(dx)^2 + (dy)^2 + (dz)^2\} - (ldx + \ldots)^2 = ds^2\,;$$

$$\therefore\ \frac{1}{\sigma} = \frac{Xd^3x + Yd^3y + Zd^3z}{X^2 + Y^2 + Z^2}.$$

If there be no tortuosity, or the curve be plane,

$$Xd^3x + Yd^3y + Zd^3z = 0$$

at every point of the curve, as in Art. 527.

Geometrical Interpretations.

541. Saint Venant observes* that, if we take three consecutive points P, Q, R for which $\xi = x$, $x + dx$, $x + 2dx + d^2x$ respectively, the projections of PQ, QR upon the axis of x will be dx, $dx + d^2x$; and if the parallelogram $PQRM$ be completed, by projecting the sides of the triangle PQM in order, since

* *Journal de l'Ecole Pol.*, vol. 18.

$PM = QR$, we shall have $dx +$ projection of $QM - (dx + d^2x) = 0$; therefore d^2x is the projection of QM.

542. In the general case, if a figure be drawn in which dx, d^2x, dy, d^2y are all positive, the projection of twice the triangle PQM on the plane of xy will be easily seen to be

$$dxd^2y + dydx - dy\,(dx + d^2x) = dxd^2y - dyd^2x.$$

Again, if Mm be drawn perpendicular to PQ, $PQ = ds$, and ultimately $mQ = QR - PQ = d^2s$;

$$\therefore\ Mm^2 = QM^2 - Qm^2 = (d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2,$$

$$\text{and } \frac{1}{\rho^2} = \text{limit of } \left(\frac{Mm}{PQ^2}\right)^2 = \frac{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2}{(ds)^4}.$$

If we make s the independent variable, this implies that $QR = PQ$, in which case QM will bisect the angle PQR and be ultimately in the direction of the principal normal, the direction cosines of which will be as $d^2x : d^2y : d^2z$.

The radius of the circle circumscribing the triangle PQR will be $\dfrac{PQ^2}{QM}$; hence, if ρ be the radius of curvature,

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2.$$

543. *Osculating plane, binormal and curvature of the helix.* In the case of the helix, Art. 497,

$$\begin{aligned}
dx &= -a \sin\theta\, d\theta, & d^2x &= -a\cos\theta\,(d\theta)^2,\\
dy &= a\cos\theta\, d\theta, & d^2y &= -a\sin\theta\,(d\theta)^2,\\
dz &= na\,d\theta, & d^2z &= 0;
\end{aligned}$$

$$\begin{aligned}
dy\,d^2z - dz\,d^2y &= na^2\sin\theta\,(d\theta)^3,\\
dz\,d^2x - dx\,d^2z &= -na^2\cos\theta\,(d\theta)^3,\\
dx\,d^2y - dy\,d^2x &= a^2\,(d\theta)^3.
\end{aligned}$$

hence the equation of the osculating plane is

$$(\xi - x)(n\sin\theta) + (\eta - y)(-n\cos\theta) + \zeta - z = 0,$$

$$\text{or } n(\xi y - \eta x) + a(\zeta - z) = 0.$$

This plane contains the point $(0, 0, z)$, and therefore the radius of the cylinder which passes through the point (x, y, z), and this radius is the principal normal.

The direction cosines of the binormal will be $\sin\alpha \sin\theta$, $-\sin\alpha \cos\theta$, $\cos\alpha$ if $\alpha = \tan^{-1}n$ be the pitch of the screw.

The measures of curvature and tortuosity are respectively $\dfrac{\cos^2\alpha}{a}$ and $\dfrac{\sin\alpha \cos\alpha}{a}$.

544. *To find the radius of curvature of a curve which is the intersection of two surfaces whose equations are given: and to express it in terms of the radii of curvature of the normal sections of the two surfaces and the angle between them, the plane of each section containing the tangent to the curve.*

Employing the same notation as in Arts. 529 and 540,

$$X = dy d^2z - dz d^2y = k^2 (EdF - FdE) = k^2 \{U\Gamma^2(\phi') - U'\Gamma^2(\phi)\},$$

and if ρ be the radius of curvature,

$$\frac{(ds)^6}{\rho^2} = X^2 + Y^2 + Z^2 = k^6 \left[(U^2 + V^2 + W^2)\{\Gamma^2(\phi')\}^2\right.$$

$$\left. - 2(UU' + VV' + WW')\Gamma^2(\phi)\Gamma^2(\phi') + (U'^2 + V'^2 + W'^2)\{\Gamma^2(\phi)\}^2\right],$$

$$\text{and } (ds)^2 = k^2 (D^2 + E^2 + F^2);$$

$$\therefore \frac{1}{\rho^2} = \frac{(U^2 + V^2 + W^2)\{\Gamma^2(\phi')\}^2 - \dots}{(D^2 + E^2 + F^2)^3}.$$

Let ω be the angle between the tangent planes to the surfaces at (x, y, z) and $P^2 \equiv U^2 + V^2 + W^2$,

$$\therefore UU' + VV' + WW' \equiv PP' \cos\omega,$$

$$\text{and } D^2 + E^2 + F^2 \equiv P^2P'^2 \sin^2\omega;$$

$$\therefore \frac{1}{\rho^2} = \frac{P^2\{\Gamma^2(\phi')\}^2 - 2PP' \cos\omega . \Gamma^2(\phi') . \Gamma^2(\phi) + P'^2\{\Gamma^2(\phi)\}^2}{P^2P'^2 \sin^2\omega\,(D^2 + E^2 + F^2)^2}. \quad (1)$$

As an example of the use of the preceding formulæ, we shall obtain the radii of curvature r, r' of the normal sections by replacing the equation of the surface $\phi' = 0$ by the equation

$$\phi'_1 = lx + my + nz - p = 0$$

of a normal plane; in which case, if D_1, E_1, F_1, and Γ_1 be the

corresponding values of D, E, F, Γ, $\Gamma_1^2(\phi_1') = 0$, and, since the normal plane contains the tangent to the curve,

$$D_1 : E_1 : F_1 = dx : dy : dz = D : E : F;$$

hence, since $\omega = \frac{1}{2}\pi$, we obtain from (1)

$$\frac{1}{r^2} = \frac{\{\Gamma_1^2(\phi)\}^2}{P^2(D_1^2 + E_1^2 + F_1^2)^2} = \frac{\{\Gamma^2(\phi)\}^2}{P^2(D^2 + E^2 + F^2)^2},$$

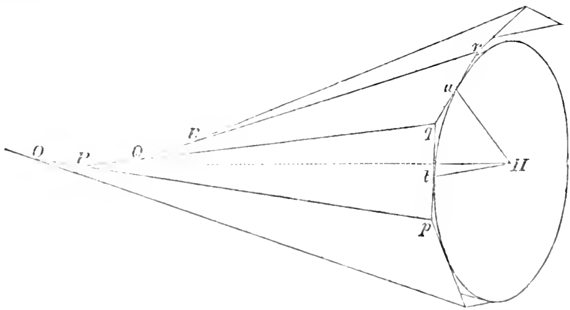
$$\text{and, similarly, } \frac{1}{r'^2} = \frac{\{\Gamma^2(\phi')\}^2}{P'^2(D^2 + E^2 + F^2)^2};$$

$$\therefore \frac{1}{\rho^2} = \frac{1}{\sin^2\omega}\left(\frac{1}{r'^2} + \frac{1}{r^2} - \frac{2\cos\omega}{rr'}\right),$$

which result will be obtained in the next chapter by Meunier's theorem.

545. *To find the vertical angle of the osculating cone of a curve.*

Let pOo, qPp, rQq be three consecutive planes which become ultimately the osculating planes of a curve $OPQR$; these planes intersect in P.



Take P as the vertex of a circular cone which touches each of the planes, this cone is in the limit, the *osculating cone* of the curve at P. Let PH be its axis, op, pq, qr the sections of the three planes made by a plane perpendicular to the axis, and t, u the points of contact with pq, qr.

Draw tH, uH perpendicular to the planes pPq, qQr; then the angle tHu will be the angle of torsion, and pPq the angle

of contingence, and we shall have $d\tau = \dfrac{2qt}{Ht}$ and $dz = \dfrac{2qt}{Pt}$ ultimately; therefore, if 2ψ be the vertical angle of the cone,

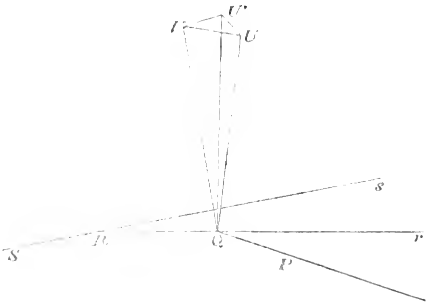
$$\tan\psi = \frac{Ht}{Pt} = \frac{dz}{d\tau} = \frac{\sigma}{\rho}.$$

546. *The rectifying line is the axis of the osculating cone at any point of the curve.*

For each of the planes through the tangent lines PQ, QR perpendicular to the osculating planes pPq, qQr ultimately contains the axis PH of the cone.

547. *To find the rate per unit of length at which the angle between principal normals increases.*

Let PQ, rQR, sRS be the directions of the sides of a polygon which are ultimately tangents to a curve.



In the planes PQr, rRs respectively draw QU, QV perpendicular to rQR, sRS, these are ultimately in the direction of consecutive principal normals.

Draw QU' in the plane QRs, perpendicular to QR, so that UQU' is the angle between the consecutive osculating planes $= d\tau$, and $U'QV$ is the angle between consecutive tangents $= dz$.

Let $d\chi$ be the angle between the lines QU, QV, and let these lines and QU' meet a sphere of radius unity, whose centre

is in Q, in U, V, U', the angle $VU'U$ is a right angle; therefore, we have ultimately

$$VU^2 = VU'^2 + U'V^2,$$

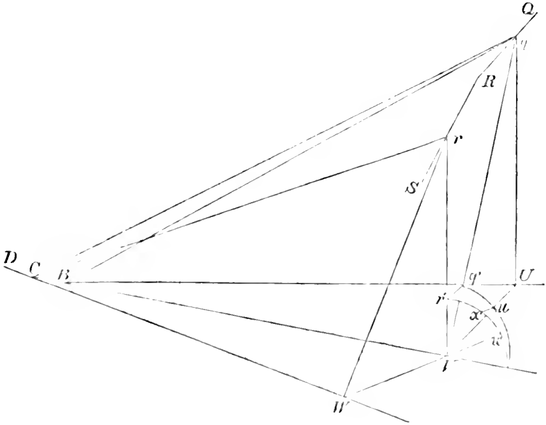
$$\text{or } (d\chi)^2 = (d\tau)^2 + (d\varepsilon)^2,$$

$\frac{d\chi}{ds}$ the rate per unit of length of the increase of the angle between principal normals is the reciprocal of what we shall call the radius of *screw*: let κ be this radius,

$$\text{then } \frac{1}{\kappa^2} = \frac{1}{\sigma^2} + \frac{1}{\rho^2}.$$

It may be noticed that $d\tau = d\chi \cos\psi$ and $d\varepsilon = d\chi \sin\psi$, which represents that the angular velocity round the rectifying line, which is normal to the plane UQV, is the resultant of two angular motions round the tangent line and the binormal which produce the rectification.

548. *To find the angle of contingence, and the element of the arc, of the locus of the centres of curvature of a given curve.*



Let QR, RS be sides of an equilateral polygon which are ultimately tangents to the given curve; let BV be the in-

tersection of planes perpendicular to QR, RS through q, r their middle points; BV is therefore ultimately a tangent to the edge of regression of the polar developable; let BU, CW be the tangents preceding and succeeding BV.

From q draw qU, qV perpendicular to BU and BV, join rV, which will be perpendicular to BV, and draw rW erpendicular to CW.

If WV be produced to w, UVw and UV will ultimately be the angle of contingence $d\varepsilon'$ and element ds' required.

Let R be the radius of spherical curvature at q, and ϕ the angle which it makes with the polar line BU. Since circles can be circumscribed about the quadrilaterals $BVUq$, $CwVr$,

$$qVU = qBU = \phi, \text{ and } VBU = VqU = d\tau, \text{ ultimately,}$$

$$\text{also } rVw = rCW = rCV + VCW = \phi + d\phi + d\tau \text{ ultimately.}$$

With V as centre and radius unity describe a sphere, and let q', r', u, w be the projections of q, r, U, w upon it, and draw ux perpendicular to $r'w$, then $q'r' = d\varepsilon$, $ux = d\varepsilon \cos\phi$, $xr' = uq' = \phi$, and $wu^2 = ux^2 + xw^2$ ultimately;

$$\therefore\ (d\varepsilon') = (d\varepsilon \cos\phi)^2 + (d\phi + d\tau)^2.$$

By drawing a diameter through V to the circle $BVUq$, it is easily seen that $VU = R \sin VBU$, and therefore that $ds' = Rd\tau$.

549. *To find the radius of spherical curvature.*

If in the figure $BVUq$, UM be drawn perpendicular to QV, $VM = d\rho$ ultimately;

$$\therefore\ (Rd\tau)^2 = (\rho d\tau)^2 + (d\rho)^2;$$

$$\therefore\ R^2 = \rho^2 + \sigma^2 \left(\frac{d\rho}{ds}\right)^2,$$

$$\text{also } Rd\tau \cos\phi = VM = d\rho;$$

therefore the distance between the centres of circular and spherical curvature $= R\cos\phi = \dfrac{d\rho}{d\tau} = \sigma\dfrac{d\rho}{ds}$.

550. *To find expressions for the radius of curvature of the edge of regression of the polar developable of a curve.*

These are readily obtained by a method suggested by Routh,* which can be explained by the last figure.

Considering the curve BCD, U and V are the feet of the perpendiculars from q on the tangents to the curve; and substituting the corresponding letters in the known formulæ $p+\dfrac{d^2p}{d\psi^2}$, $r\dfrac{dr}{dp}$, we obtain two expressions for the radius of curvature of the edge of regression, viz. $\rho+\dfrac{d^2\rho}{d\tau^2}$ and $R\dfrac{dR}{d\rho}$.

From the expression $\dfrac{dp}{d\psi}$ for the distance of the foot of the perpendicular from the point of contact, we obtain

$$R\cos\phi=\frac{d\rho}{d\tau}, \text{ as in Art. 549.}$$

It will repay the student to read the paper referred to.

551. *To find the coordinates of any point of a curve in terms of the arc, when the axes of coordinates are the tangent, principal normal, and binormal at the point from which the arc is measured.*

Let Ox be the tangent at O, Oy the principal normal, Oz the binormal, the planes of yz, zx, xy being the normal, rectifying, and osculating planes, and let s be the distance of a point (x, y, z) from O, measured along the arc.

Then, at the origin,

$$\frac{dx}{ds}=1,\quad \frac{dy}{ds}=0,\quad \frac{dz}{ds}=0,$$

$$\rho\frac{d^2x}{ds^2}=0,\quad \rho\frac{d^2y}{ds^2}=1,\quad \rho\frac{d^2z}{ds^2}=0,$$

these quantities being the direction cosines of the tangent and principal normal. If α be the angle made by the tangent at (x, y, z) with the tangent Ox, $\dfrac{dx}{ds}=\cos\alpha$;

$$\therefore\ \frac{d^3x}{ds^3}=-\cos\alpha\left(\frac{d\alpha}{ds}\right)^2-\sin\alpha\frac{d^2\alpha}{ds^2};$$

therefore, at O, since $d\alpha$ is the angle of contingence, and $\alpha=0$,

$$\frac{d^3x}{ds^3}=-\frac{1}{\rho^2}.$$

* *Quarterly Journal*, vol. VII., p. 42.

Again, if β be the angle made by the principal normal at (x, y, z) with Oy,

$$\rho \frac{d^2y}{ds^2} = \cos\beta; \quad \therefore \frac{d\rho}{ds}\frac{d^2y}{ds^2} + \rho\frac{d^3y}{ds^3} = -\sin\beta\frac{d\beta}{ds};$$

therefore, at O,

$$\frac{1}{\rho}\frac{d\rho}{ds} + \rho\frac{d^3y}{ds^3} = 0;$$

and if γ be the angle made by the binormal at (x, y, z) with the principal normal at the origin,

$$\rho\left(\frac{dz}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2z}{ds^2}\right) = \cos\gamma;$$

$$\therefore \frac{d\rho}{ds}\left(\frac{dz}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2z}{ds^2}\right) + \rho\left(\frac{dz}{ds}\frac{d^3x}{ds^3} - \frac{dx}{ds}\frac{d^3z}{ds^3}\right) = -\sin\gamma\frac{d\gamma}{ds};$$

therefore, at O, $\rho\frac{d^3z}{ds^3} = \sin\gamma\frac{d\gamma}{ds} = \frac{1}{\sigma}$, since $d\gamma$ is the angle of torsion; therefore, by Maclaurin's theorem,

$$x = s - \frac{s^3}{6\rho^2},$$

$$y = \frac{s^2}{2\rho} - \frac{s^3}{6\rho^2}\frac{d\rho}{ds},$$

$$z = \frac{s^3}{6\rho\sigma}.$$

552. *To find the angle between two consecutive principal normals.*

The direction cosines of the principal normal at (x, y, z), a point near the origin, are as $-\frac{s}{\rho^2} : \frac{1}{\rho} : \frac{s}{\rho\sigma}$, and the secant of the angle between this normal and that at the origin is $\sqrt{\left\{1 + s^2\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)\right\}}$; therefore, the angle required is ultimately $s\sqrt{\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)}$;

$$\therefore \frac{1}{\kappa^2} = \frac{1}{\rho^2} + \frac{1}{\sigma^2},$$

where κ is the radius of screw, as in Art. 547.

553. *To find the shortest distance between consecutive principal normals and its position.*

The equations of a principal normal at (x, y, z) are approximately

$$-\rho(\xi - x) = s(\eta - y) = \sigma(\zeta - z),$$

and its shortest distance from the principal normal at the origin is the perpendicular distance from the origin upon its projection on the plane of zx, whose equation is

$$\rho(\xi - s) + \sigma\zeta = 0,$$

hence the shortest distance is $\dfrac{\rho s}{\sqrt{(\rho^2 + \sigma^2)}} = \dfrac{\kappa s}{\sigma}$.

If the line on which the shortest distance lies meet the axis of y at a distance ρ' from the origin, the equations of the line will be $\dfrac{\xi}{\rho} = \dfrac{\eta - \rho'}{0} = \dfrac{\zeta}{\sigma}$, and this line will intersect the line $-\rho(\xi - s) = s\eta = \sigma\zeta$;

$$\therefore \frac{\rho - \rho'}{\rho^2} = \frac{\rho'}{\sigma^2} = \frac{\rho}{\rho^2 + \sigma^2}.$$

Hence the shortest distance divides the radius of curvature into two segments which are in the duplicate ratio of the radii of curvature and torsion.

554. *To find the angle between the rectifying line and the tangent at any point.*

The tangent plane to the rectifying developable at any point contains the tangent and binormal, and its normal is the principal normal whose direction cosines are in the ratio $-\dfrac{s}{\rho^2} : \dfrac{1}{\rho} : \dfrac{s}{\rho\sigma}$, retaining only the principal terms.

Therefore the rectifying line is the ultimate intersection of the planes $\eta = 0$, and

$$-\frac{s}{\rho}(\xi - x) + \eta - y + \frac{s}{\sigma}(\zeta - z) = 0.$$

Hence the tangent of the angle made with the tangent to the curve at O is $\dfrac{\sigma}{\rho}$.

555. *To find the element of the arc of the locus of the centres of curvature.*

If ξ, η, ζ be the coordinates of the centre of curvature at (x, y, z), neglecting s^2,

$$\frac{\xi - x}{-\frac{s}{\rho}} = \frac{\eta - y}{1} = \frac{\zeta - z}{\frac{s}{\sigma}} = \rho + \frac{d\rho}{ds} s,$$

and the element of the arc is ultimately

$$\sqrt{\{\xi^2 + (\eta - \rho)^2 + \zeta^2\}} = s\sqrt{\left\{\left(\frac{d\rho}{ds}\right)^2 + \left(\frac{\rho}{\sigma}\right)^2\right\}}.$$

The direction cosines of the tangent to the locus are as

$$\xi : \eta - \rho : \zeta = 0 : \frac{d\rho}{ds} : \frac{\rho}{\sigma}.$$

556. *To find the axis and pitch of the helix which has the same curvature and tortuosity as a given curve.*

Let a be the radius of the cylinder and α the pitch, then, referred to the tangent, principal normal, and binormal, the coordinates corresponding to an arc s are, by transformation of coordinates,

$$x = a \cos\alpha \sin\left(\frac{s \cos\alpha}{a}\right) + s \sin^2\alpha,$$

$$y = a - a \cos\left(\frac{s \cos\alpha}{a}\right),$$

$$z = -a \sin\alpha \sin\left(\frac{s \cos\alpha}{a}\right) + s \sin\alpha \cos\alpha,$$

and, equating these coordinates to those of the curve as far as s^3,

$$s - \frac{\cos^4\alpha}{6a^2} s^3 = s - \frac{s^3}{6\rho^2},$$

$$\frac{\cos^2\alpha}{2a} s^2 = \frac{s^2}{2\rho} - \frac{s^3}{6\rho^2}\frac{d\rho}{ds},$$

$$\frac{\cos^3\alpha \sin\alpha}{6a^2} s^3 = \frac{s^3}{6\rho\sigma};$$

$$\therefore\ a = \rho \cos^2\alpha = \sigma \sin\alpha \cos\alpha,$$

$$\frac{\cos\alpha}{\sigma} = \frac{\sin\alpha}{\rho} = \frac{1}{\sqrt{(\rho^2 + \sigma^2)}} \text{ and } a = \frac{\rho\sigma^2}{\rho^2 + \sigma^2};$$

hence the axis, whose inclination to the binormal is $\alpha = \tan^{-1}\frac{\rho}{\sigma}$, lies along the shortest distance between consecutive principal normals.

Also, if along a curve and the osculating helix equal small arcs be measured from the point of contact and on the same side of it, the distance between the ends of these arcs will be ultimately $\frac{s^3}{6\rho^2}\frac{d\rho}{ds}$.

Line of Greatest Slope.

557. Def. The line of greatest slope on a surface is the curve which at every point is inclined at a greater angle to a given plane than any other line drawn through that point on the surface.

If the given plane be horizontal, the bed of a shallow brook on a hill side will be a line of greatest slope which the water will have selected for its course.

558. *To find the equations of the line of greatest slope on a given surface.*

Let $F(\xi, \eta, \zeta) = 0$ be the equation of the surface, l, m, n the direction-cosines of the given plane.

The equation of the tangent plane at any point (x, y, z) is

$$U(\xi - x) + V(\eta - y) + W(\zeta - z) = 0.$$

The direction-cosines of the line in which this plane cuts the given plane are proportional to

$$mW - nV,\ \ nU - lW,\ \ lV - mU.$$

Of all the tangent lines drawn through (x, y, z), that line which has the greatest inclination to the given plane is perpendicular to the line of intersection.

Hence the differential equation

$$(mW - nV)\,dx + (nU - lW)\,dy + (lV - mU)\,dz = 0,$$

with the equation of the surface, determines all the lines of greatest slope which can be drawn on the surface, the constant introduced in the integration being determined for any particular curve by some point through which it passes.

If the given plane be the plane xy, since $l=0$, $m=0$, the equations of the lines of greatest slope will be

$$Vdx - Udy = 0 \text{ and } F(x, y, z) = 0,$$

perpendicular to the lines of level $Udx + Vdy = 0$.

559. *To find the lines of greatest slope on the cone* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$, *the plane xy being the plane of reference.*

The differential equation is, in this case,

$$\frac{y}{b^2}\,dx - \frac{x}{a^2}\,dy = 0;$$

$$\therefore\ a^2 \log x - b^2 \log y = \log C;$$

$$\therefore\ x^{a^2} = Cy^{b^2}.$$

The lines of greatest slope are the intersection of the cone with the cylinders represented by this equation; and it may be observed that no generating line, except those in the principal planes, is a line of greatest slope, unless the cone be a right cone.

Line, Surface, and Volume-Integral.

560. We give here two theorems relating to line, surface, and volume-integrals, which are of great importance in certain problems in Electricity and Conduction of Heat, and which serve as illustrations of the subjects of this and the preceding chapter.

Line-Integral and Surface-Integral.

561. DEF. If R be any quantity having direction, called a vector quantity, and ε be the angle between its direction and that of the tangent to a curve at any point (x, y, z) taken in a definite direction, the integral $\int R\cos\varepsilon\, ds$ is called the *line-integral* of R along the line s, s being measured from a fixed point. The integral may be written $\int\left(u\frac{dx}{ds} + v\frac{dy}{ds} + w\frac{dz}{ds}\right)ds$, u, v, w being functions of x, y, z.

If η be the angle between the direction of R and a normal to a surface at any point (x, y, z), the integral $\iint R\cos\eta\, dS$ is called the *surface-integral*, the summation being taken over

the whole of a surface S. The integral may be written $\iint(Ul+Vm+Wn)\,dS$, U, V, W being functions of (x, y, z), and l, m, n the direction-cosines of the normal to the surface at (x, y, z) measured in a definite direction.

562. *To shew that a line-integral taken round a given closed curve can be represented as a surface-integral of a surface bounded by the given curve.*

Suppose the closed curve L to be filled up by any surface S, and suppose S to be divided into an infinite number of small elements, one of which is σ bounded by the line λ. If we take the sum of the line-integrals for two of these lines which have a common part μ, both estimated in the same direction, the two portions of the sums taken over μ will be taken in opposite directions, and being of the same magnitude will vanish; those lines λ which abut upon L are the only portions which will not be traversed twice; hence the sum of all the line-integrals for the elements λ will be that of the line L.

The proposition will, therefore, be proved if we shew that it is true for any elementary line λ and corresponding surface σ.

Let (x, y, z) be any point on σ, and $(x+\xi, y+\eta, z+\zeta)$ any point on λ; the line-integral for λ, u, v, w being given at (x, y, z), $= \int\left\{\left(u+\frac{du}{dx}\xi+\frac{du}{dy}\eta+\frac{du}{dz}\zeta\right)d\xi+\ldots\right\}$ ultimately.

Since λ is a closed curve, $\int d\xi=0$, $\int\xi d\xi=0$, and if we suppose the summation taken in the direction from x to y,

$$-\int\eta d\xi=\int\xi d\eta=n\sigma\,;$$

hence the line-integral for λ is $\left\{\left(\frac{dv}{dx}-\frac{du}{dy}\right)n+\ldots\right\}\sigma$.

The line-integral of L is, therefore, equal to the surface-integral $\iint(Ul+Vm+Wn)\,ds$, when $U=\frac{dw}{dy}-\frac{dv}{dz}$, &c.

563. *The surface-integral of a directed quantity or vector, taken over a closed surface, may be expressed as a volume-integral of a certain function.*

We observe that if the theorem can be proved for an elementary portion of the volume enclosed within the surface,

within which the directed quantity is supposed to be continuous, the general theorem will follow, as well as its modification, when the enclosed volume is intersected by surfaces across which the directed quantity changes discontinuously.

For, if v_1, v_2 be two elementary volumes enclosed by the surfaces σ_1, σ_2 to which a portion σ' is common, the normal components along σ', which belong respectively to σ_1 and σ_2, being in opposite directions and of the same magnitude, will disappear in the summation.

If, therefore, we sum for all elements within a volume V, throughout which the value of the vector changes continuously, the only points of the resolved vectors which are not destroyed are those which belong to the points of the elements which abut on the enclosing surface S.

If the vector change discontinuously in passing surfaces Σ_1, Σ_2, &c., the theorem will hold for the portions V_1, V_2... into which they divide V, and the volume-integral over V would be equal to the surface-integral over S, together with the surface-integrals over Σ_1, Σ_2; the differences of the vectors on opposite sides of these surfaces replacing the vectors in the first integral.

Let an elementary volume v be inclosed by the surface σ, (x, y, z) being any point within v, and let $(x+\xi, y+\eta, z+\zeta)$ be any point upon σ, and u, v, w the components of the vector at (x, y, z) parallel to the axes.

The surface-integral for σ is

$$\iint\left\{\left(u+\frac{du}{dx}\xi+\ldots\right)l+\left(v+\frac{dv}{dy}\eta+\ldots\right)m+\ldots\right\}d\sigma,$$

$$\iint l\,d\sigma=0,\quad \iint\xi l\,d\sigma=\iint\xi\,d\eta\,d\zeta=v,\ \&c.;$$

hence the surface-integral for $\sigma=\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)v$ ultimately;

therefore, if u_1, u_1' be the values of u on opposite sides of Σ_1,

$$\iint(ul+vm+wn)\,dS=\iiint\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)dV$$
$$+\iint\{(u_1-u_1')\,l'+(v_1-v_1')\,m'+(w_1-w_1')\,n'\}\,d\Sigma_1$$
$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;$$

which represents the theorem in its most general form.

XX.

(1) The equations of the tangent to the curve of intersection of the surfaces $ax^2 + by^2 + cz^2 = 1$ and $bx^2 + cy^2 + az^2 = 1$ are

$$\frac{x(\xi - x)}{ab - c^2} = \frac{y(\eta - y)}{bc - a^2} = \frac{z(\zeta - z)}{ac - b^2}.$$

The tangent line at the point $x = y = z$ lies in the plane

$$(a - b)x + (b - c)y + (c - a)z = 0.$$

If $ac = b^2$, the tangent lines trace on the plane of xy the two straight lines whose equation is $\dfrac{cx^2}{c^3 - b^3} = \dfrac{ay^2}{a^3 - b^3}$.

(2) The equations of a sphere and cylinder being

$$x^2 + y^2 + z^2 = 4a^2 \text{ and } x^2 + z^2 = 2ax,$$

prove that the equations of the tangent to the curve of intersection at the point (α, β, γ) are

$$(\alpha - a)x + \gamma z = a\alpha \text{ and } \beta y + ax = a(4a - \alpha),$$

and that the equation of the normal plane is

$$\frac{x}{\alpha} = \frac{y}{\beta} = \left(1 - \frac{a}{\alpha}\right)\left(\frac{z}{\gamma} - \frac{y}{\beta}\right).$$

(3) Find the tangent line of the intersection of the surfaces

$$z(x + z)(x - a) = a^3 \text{ and } z(y + z)(y - a) = a^3,$$

and shew that it consists of plane curves.

(4) The paraboloid whose equation is $ax^2 + by^2 = 4z$ has traced upon it a curve, every point of which is the extremity of the latus rectum of the parabolic section through the axis of z; shew that the tangent to the curve traces upon the plane of xy the curves whose equations are

$$r \sin 2\theta = \pm (a \sim b).$$

(5) Shew that the equation of the normal plane at any point f, g, h on the curve defined by the equations $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, $x^2 + y^2 + z^2 = d^2$ is

$$x\,\frac{a^2(b^2 - c^2)}{f} + y\,\frac{b^2(c^2 - a^2)}{g} + z\,\frac{c^2(a^2 - b^2)}{h} = 0.$$

(6) A point moves on an ellipsoid so that its direction of motion always passes through the perpendicular from the centre of the ellipsoid on the tangent plane at the point; shew that the curve traced out by the point is given by the intersection of the ellipsoid with the surface

$$x^{m-n}y^{n-l}z^{l-m} = \text{constant},$$

l, m, n being inversely proportional to the squares on the semi-axes of the ellipsoid.

(7) If the osculating plane at every point of a curve pass through a fixed point, prove that the curve is plane. Hence prove that the curves of intersection of the surfaces whose equations are

$$x^2 + y^2 + z^2 = a^2 \text{ and } x^4 + y^4 + z^4 = \tfrac{1}{2}a^4$$

are circles of radius a.

(8) Prove that if every pair of consecutive principal normals to a curve intersect, the curve must be a plane; and find $f(\theta)$ so that the curve, whose coordinates are given by $x = a \cos\theta$, $y = b \sin\theta$, $z = f(\theta)$, may be plane.

(9) A curve is traced on a right cone so as always to cut the generating line at the same angle; shew that its projection on the plane of the base is an equiangular spiral.

(10) If a string be unwound from a helix so that the straight portion is a tangent to it, shew that any point on the string will describe the involute of a circle.

(11) Prove that the locus of the centres of curvature of a helix is a similar helix; and find the condition that it shall be traced on the same cylinder.

(12) When the radius of curvature is a maximum or minimum, the tangent to the locus of the centres of curvature is perpendicular to the radius of curvature.

(13) Coordinates of any point in a curve are $x = 4a \cos^3\theta$, $y = 4a \sin^3\theta$, $z = 3c \cos^2\theta$; find the equations of the normal and osculating planes; and find the relation between c and a that the locus of the centre of the sphere of curvature may be a curve similar to the original curve.

(14) A straight line is drawn on a plane, which is then wrapped in a cone; shew that the radius of curvature of the curve on the cone varies as the cube of the distance from the vertex.

(15) If a tortuous curve be projected on a plane, the normal to which is inclined at angles α, β to the tangent and binormal at any point, the curvature of the projection will be to that of the curve as $\cos\gamma : \sin^2\alpha$.

(16) If ρ be the radius of curvature of a curve, then that of its projection on a plane inclined at an angle α to the osculating plane is $\rho \sec\alpha$ if the plane be parallel to the tangent, and $\rho \cos^2\alpha$ if it be parallel to the principal normal.

(17) If the measures of curvature and tortuosity of a curve be constant at every point of a curve, the curve will be a helix traced on a cylinder.

(18) If $\dfrac{1}{\rho}$, $\dfrac{1}{\sigma}$ be the measures of curvature and tortuosity at any point of a curve in space; ρ', σ' similar quantities at the corresponding point of the locus of the centre of spherical curvature; then $\sigma\sigma' = \rho\rho'$.

(19) Prove that the equation of the polar surface to the helix is

$$x \cos\phi + y \sin\phi + a \tan^2\alpha = 0,$$

$$\text{where } x^2 + y^2 = \tan^2\alpha \, \{a^2 \tan^2\alpha + (z - a \tan\alpha\, \phi)^2\},$$

and that its edge of regression is a helix on a cylinder whose radius is $a \tan^2\alpha$.

(20) Prove that the angle between the radius of the osculating sphere and the edge of regression of the polar surface is equal to the angle between the radius of the osculating circle and the locus of the centre of curvature.

(21) When the polar surface of a curve is developed into a plane, prove that the curve itself degenerates into a point on the plane; and if r, p be the radius vector and perpendicular on the tangent to the developed edge of regression of the polar surface drawn from this point, prove that ρ, σ and s referring to the points on the original curve, $\sqrt{(r^2 - p^2)} = \sigma \dfrac{d\rho}{ds}$.

(22) Shew that the shortest distance of tangents at the extremity of a small arc δs of a helix, whose pitch is α and radius of cylinder a, is

$$\frac{\sin\alpha \cos^3\alpha}{12a^2} (\delta s)^3.$$

(23) Prove that the angle between the shortest distance of tangents at two consecutive points and the binormal at one of them is equal to half the corresponding angle of torsion.

(24) The equations of a curve are

$$x = \frac{s}{\sqrt{6}} \cos\left(\frac{1}{\sqrt{2}} \log \frac{2s^2}{3s^4}\right), \quad y = \frac{s}{\sqrt{6}} \sin\left(\frac{1}{\sqrt{2}} \log \frac{2s^2}{3c^2}\right), \quad z = \frac{s}{\sqrt{2}},$$

prove that the curvature and tortuosity at any point are each equal to $\dfrac{1}{s}$.

(25) A spiral is traced on a hemisphere, such that, if l, λ be the longitude and latitude of any point, $4\lambda + l = 2\pi$; shew that the surface of the hemisphere is divided by the spiral in the ratio $\pi - 2 : 2$.

(26) A curve is formed by the intersection of a hemisphere, and a cylinder whose base is the circle described on a radius of the base of the hemisphere as diameter; prove that the portion of the area of the hemisphere included between the curve, the meridian touching the cylinder, and a quadrant of the base of the hemisphere, is equal to the square on the radius of the hemisphere.

(27) Prove that the volume contained between the cylinder, the hemiphere, the meridian plane touching the cylinder, and the base of the shemisphere, is $\frac{2}{9}$ths of the cube of the radius of the hemisphere.

(28) A hemisphere is pierced by a cylinder, whose circular base touches the base of the hemisphere, the diameter of the base of the cylinder being less than the radius of the hemisphere; prove that the area of the cylinder included between the hemisphere and its base is equal to the rectangle contained by the diameter of the cylinder and the chord of the base of the hemisphere which touches the base of the cylinder and is parallel to the common tangent of the bases.

(29) Find the equation of the projection on the plane of xy of the lines of greatest slope on the surface $z = x + \frac{1}{2}c \log \frac{x^2+y^2}{a^2}$, the plane of xy being horizontal.

(30) Prove the following equations of the lines of greatest slope on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, placed in any position,

$$p\frac{dx}{ds} + a^2\frac{d\psi}{dx} = 0, \quad p\frac{dy}{ds} + b^2\frac{d\psi}{dy} = 0, \quad p\frac{dz}{ds} + c^2\frac{d\psi}{dz} = 0,$$

where ψ is the inclination of the normal to the vertical at the point (x, y, z), and p the perpendicular on the tangent plane at that point.

(31) If x, y, z be the coordinates of any point of a sphere, ds an element of an arc of any curve on the sphere passing through x, y, z, and if

$$y\frac{dz}{ds} - z\frac{dy}{ds} = \xi, \quad z\frac{dx}{ds} - x\frac{dz}{ds} = \eta, \quad x\frac{dy}{ds} - y\frac{dx}{ds} = \zeta,$$

and $d\xi^2 + d\eta^2 + d\zeta^2 = d\sigma^2$,; prove that

$$\eta\frac{d\zeta}{d\sigma} - \zeta\frac{d\eta}{d\sigma} = x, \quad \zeta\frac{d\xi}{d\sigma} - \xi\frac{d\zeta}{d\sigma} = y, \quad \xi\frac{d\eta}{d\sigma} - \eta\frac{d\xi}{d\sigma} = z.$$

If the locus of (x, y, z) be a closed curve, prove that the solid angle which it subtends at the centre of the sphere together with the perimeter of the locus of (ξ, η, ζ) is equal to 2π, and *vice versâ*.

(32) If a surface and a cone whose vertex is at the origin be referred to polar coordinates, the area of the cone between two of its generators and the curve in which it meets the surface is $\frac{1}{2}\int r^2 \left\{1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2\right\}^{\frac{1}{2}} d\theta$.

The equations of a cylinder and cone are

$$r\sin\theta = a \quad \text{and} \quad \cot\theta = \tfrac{1}{2}(e^{\phi} - e^{-\phi}).$$

If A_1, A_2, and A_3 be the areas of the cone reckoned from $\phi = 0$ to $\phi = \beta - \alpha$, β and $\beta + \alpha$ respectively; then will $A_1 + A_3 = (e^{\alpha} + e^{-\alpha}) A_2$.

CHAPTER XXI.

CURVATURE OF SURFACES. NORMAL SECTIONS. INDICATRIX. DISTRIBUTION OF NORMALS. SURFACE OF CENTRES. INTEGRAL CURVATURE. DIFFERENTIAL EQUATION OF LINES OF CURVATURE. UMBILICS.

564. In this chapter we proceed to examine the curvature of surfaces, and to explain how the amount of curvature may be estimated.

If the student will consider the simpler surfaces with which he is familiar, such as a sphere, an ellipsoid, or an hyperboloid of one sheet, he will have examples of the kind of flexure which may take place at an ordinary point of any surface.

Any point of a sphere or a pole of a prolate or of an oblate spheroid is an example of a point of a surface from which, if we proceed along any section made by a plane containing the normal; the curvature is the same.

Any point of an ellipsoid is an example of a point on a surface at which, if a tangent plane be drawn, the surface in the neighbourhood of the point lies entirely on one side of the tangent plane; such surfaces are called *Synclastic*.

If a tangent plane be drawn at any point of an hyperboloid of one sheet, the surface will intersect the tangent plane, and bend from it partly on one side and partly on the other; such surfaces are called *Anticlastic*.

565. Let two planes be drawn through the same tangent line at any point of a sphere, one containing the normal and the other inclined at an angle θ to the normal, the sections made by these planes will have their radii in the ratio $1 : \cos\theta$.

This simple relation between the radii of curvature of a normal and oblique section, containing the same tangent line, will be proved to be true for any surface at an ordinary point.

The student may for an exercise prove it when the tangent line is drawn through the extremity of a principal axis of an ellipsoid parallel to another principal axis.

566. Consider next the curvature of the sections of an ellipsoid made by planes passing through OA, the normal at A, an extremity of one of the principal axes; if AP be one of these sections intersecting the principal section BC, perpendicular to OA, in P; OP and OA will be its semiaxes, and its radius of curvature at $A = \dfrac{OP^2}{OA}$; also $\dfrac{OB^2}{OA}$, $\dfrac{OC^2}{OA}$ will be the radii of curvature at A of the principal sections BA and CA, and since OB, OP, OC are in order of magnitude for all positions of OP, we see that of all the normal sections through A, the two sections which have their curvature a maximum and minimum have their planes perpendicular to each other.

This property of normal sections will be found to hold for any ordinary point of all surfaces.

567. If $POB=\theta$, $\dfrac{\cos^2\theta}{OB^2} + \dfrac{\sin^2\theta}{OC^2} = \dfrac{1}{OP^2}$; hence, if ρ, ρ' be the radii of curvature of the sections AB, AC, and R that of the section AP, $\dfrac{\cos^2\theta}{\rho} + \dfrac{\sin^2\theta}{\rho'} = \dfrac{1}{R}$.

This relation between the radii of curvature of the normal sections of least and greatest curvature, called principal sections, and that of any other normal section inclined at an angle θ to one of the principal sections will also be found to hold for any surface.

568. These three properties which are true for all surfaces will enable us to determine the radii of curvature of all plane sections through any point, when those of any two sections, not containing the same tangent, are known.

Normal Sections.

569. *To find the relation between the radii of curvature of sections made by planes containing the normal at any point of a surface.*

Let the surface be referred to the normal at the point O as the axis of z, and the tangent plane at O as the plane of xy.

The values of $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at the origin vanish, and, if r, s, t be the values of $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dxdy}$, $\frac{d^2z}{dy^2}$,

$$z = \tfrac{1}{2}(rx^2 + 2sxy + ty^2) + \&c.$$

Let the surface be cut by a plane passing through Oz, and inclined at an angle θ to the plane of zx; at every point of this plane $x = u\cos\theta$, $y = u\sin\theta$;

$$\therefore\ z = \tfrac{1}{2}(r\cos^2\theta + 2s\cos\theta\sin\theta + t\sin^2\theta)\,u^2(1+\varepsilon)$$

where ε vanishes in the limit.

If R be the radius of curvature of this section,

$$\frac{1}{R} = \text{limit}\ \frac{2z}{u^2} = r\cos^2\theta + 2s\cos\theta\sin\theta + t\sin^2\theta.$$

The directions of the normal sections of which the curvature is a maximum or minimum are given by the equation

$$-(r-t)\sin 2\theta + 2s\cos 2\theta = 0.$$

If α be one solution, the rest will be included in the formula $\frac{1}{2}n\pi + \alpha$, hence the sections of maximum and minimum curvature are at right angles.

These sections are called the *Principal Sections* of the surface at the point considered.

If the planes of the principal sections be taken for the planes of zx and yz, $\alpha = 0$, and therefore $s = 0$, and the expression for the curvature of any section will become $\frac{1}{R} = r\cos^2\theta + t\sin^2\theta$; let ρ, ρ' be the radii of curvature of the principal sections, then $\frac{1}{\rho} = r$ and $\frac{1}{\rho'} = t$,

$$\therefore\ \frac{1}{R} = \frac{\cos^2\theta}{\rho} + \frac{\sin^2\theta}{\rho'};$$

also, if R, R' be the radii of curvature of any perpendicular normal sections,

$$\frac{1}{R} + \frac{1}{R'} = \frac{1}{\rho} + \frac{1}{\rho'}.$$

These theorems are due to Euler.

The Indicatrix.

570. Euler's theorems and other theorems relating to the curvature of plane sections of surfaces are deduced with great facility by means of a curve called the indicatrix, employed first by Dupin for this purpose.

Def. The *indicatrix* at any point P of a surface is the section made by a plane parallel to the tangent plane at P and at an infinitely small distance from it.

In cases in which, as in anticlastic surfaces, the curve of intersection extends to any finite distance, the name of indicatrix only applies to the portions of the curve which are infinitely near to P.

571. *At any ordinary point of a surface the indicatrix is a conic.*

Taking the axes as in Art. 569, the equation of the surface is of the form $z = \frac{1}{2}(rx^2 + 2sxy + ty^2) + \&c.$, and by transformation of axes the term involving xy may be made to disappear, so that $z = ax^2 + by^2 +$ terms of higher dimensions.

If the surface be cut by a plane parallel to the tangent plane and very near to it, for which $z = h$, in the neighbourhood of the point of contact $h = ax^2 + by^2$; the indicatrix is therefore a conic whose centre is in the normal.

572. Pendlebury has noticed* that the indicatrix may be, at particular points of some surfaces, of any form, and the number of directions of principal curvature for such points may be any number, in fact, equal to the number of apses in the indicatrix. He gives as an example a surface $x^2 + y^2 = az\phi\left(\frac{y}{x}\right)$ generated by a parabola revolving round its axis, its latus rectum increasing or decreasing with the angle through which its plane has revolved; such surface would look like a paraboloid with ridges and furrows radiating from the vertex.

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* *Messenger of Mathematics*, vol. I., p. 148.

573. *The radius of curvature of a normal section of a surface varies as the square of the corresponding central radius of the indicatrix.*

Let CP be the central radius of the indicatrix which lies in any normal section whose radius of curvature at O is R; then $2R = \text{limit}\ \dfrac{CP^2}{OC}$, (see figure, p. 386); therefore, since OC is constant for all normal sections $R \propto CP^2$.

Hence all theorems in central conics which can be expressed by equations homogeneous in terms of the radii and axes, can be replaced by corresponding theorems in curvature. Euler's theorems follow at once, and if R, R' be radii of curvature of normal sections inclined to a principal section of a surface at angles θ, θ', such that $\tan\theta \tan\theta' = -\dfrac{\rho'}{\rho}$,

$$\text{then } R + R' = \rho + \rho',$$

$$\text{and } RR' \sin^2(\theta' - \theta) = \rho\rho'.$$

574. When the indicatrix is an ellipse, the surface is synclastic at the point considered.

A point of a surface for which the indicatrix is a circle is called an *umbilic*, the curvatures of all sections made by planes containing the normal at an umbilic being equal.

575. When the form of the indicatrix is hyperbolic, the surface is anticlastic at the point considered; in this case the radii of curvature of normal sections containing the asymptotes are infinite, such sections pass through the inflexional tangents, and their directions are given by $\tan^2\theta = \dfrac{\rho'}{\rho}$, ρ, ρ' being the absolute values of the radii of curvature.

In order to deduce theorems from geometrical properties of the hyperbola, it may be necessary to suppose two indicatrices, one on each side of the tangent plane at equal distances from it.

If $\rho' = \rho$ and R be the radius of curvature of a normal section inclined at an angle θ to a principal section

$$R \cos 2\theta = \rho.$$

576. When the section made by the plane parallel to the tangent plane is a parabola, the part of the section which is called the indicatrix is two parallel lines which become ultimately, as in the case of a developable surface, two coincident lines.

Such points are called *Parabolic points*, sometimes also *Cylindrical points.*

As an example of a parabolic point, take a point of the cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$, at a distance l from the vertex in the generator $\frac{x}{a}=\frac{z}{c}$; transform the axes so that the normal at this point is the axis of z, and the generator the axis of x, the resulting equation of the cone is $lz=\frac{ac}{2b^2}y^2-zx-\frac{c^2-a^2}{2ac}z^2$, let $z=h$ and $a=c\tan\alpha$, then $y^2=\frac{2b^2}{ac}h(x+l-h\cot 2\alpha)$, the section by a plane parallel to the tangent plane is therefore a parabola, the distance of whose vertex from the normal at the point considered is $l-h\cot 2\alpha$, and since this remains finite, when h is made indefinitely small, the degeneration into two nearly coincident parallel lines in the neighbourhood of the point is explained.

The finite principal radius of curvature is $\frac{b^2l}{ac}$.

577. *The intersection of two consecutive tangent planes and the direction of the line joining the points of contact are parallel to conjugate diameters of the indicatrix.*

Let CP, CD be conjugate semi-diameters of the indicatrix for the point O of a surface; since the tangent plane to the surface at P contains the tangent to the indicatrix at P, its intersection with the tangent plane at O is parallel to CD, and proceeding to the limit, when OC vanishes, the proposition follows.

DEF. Tangent lines at any point of a surface drawn parallel to conjugate diameters of the indicatrix, are called *Conjugate Tangents.*

578. It follows from this property of consecutive tangent planes, that if a torse envelope any surface the directions of the generating lines at any point of the curve of contact are conjugate to the tangents to the curve.

579. *To find the relation between the radii of curvature of a normal and oblique section of a surface made by planes passing through the same tangent line.*

Let the tangent line through which the planes are taken be the axis of x, and let θ be the inclination of the planes of the oblique and normal sections through Ox.

The equation of the indicatrix is of the form

$$2h = ax^2 + 2cxy + by^2,$$

and where the oblique section cuts the indicatrix $y = h\tan\theta$, therefore xy and y^2 vanish, compared with x^2; hence the radius of curvature of the oblique section at O is the limit of $\dfrac{x^2}{2h\sec\theta}$, and if R, R' be the radii of curvature of the normal and oblique sections

$$R' = R\cos\theta.$$

This is Meunier's theorem.

580. Bezant* gives the following elegant proof of Meunier's theorem: Take a normal and oblique section at any point of a surface, the two curves of section having the same tangent line, and therefore having two consecutive points in common. In each of the curves take a third consecutive point and describe a sphere through the four contiguous points, the sections of the sphere by the two planes are evidently the circles of curvature of the normal and oblique sections, and the theorem follows immediately.

581. *Radius of curvature of the curve of intersection of two surfaces.*

Let ρ be the radius of curvature of the curve at any point P, r, r' those of the normal sections of the surfaces made by planes

* *Quart. Jour. of Math.*, vol. VI., p. 140.

containing the tangent at P; let ω be the angle between the planes and ϕ, $\omega-\phi$ the angles between the osculating plane of the curve at P and the two normal planes.

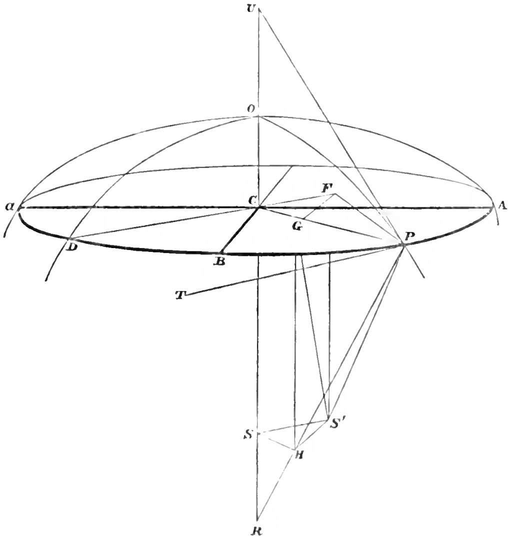
Now the curvature of the curve is the same as that of the section of either surface by the osculating plane, since they have three consecutive points in common, and by Meunier's theorem,

$$\frac{1}{r}=\frac{1}{\rho}\cos\phi, \text{ and } \frac{1}{r'}=\frac{1}{\rho}\cos(\omega-\phi)=\frac{1}{r}\cos\omega+\frac{1}{\rho}\sin\omega\sin\phi;$$

$$\therefore\ \frac{\sin^2\omega}{\rho^2}=\frac{1}{r^2}-\frac{2\cos\omega}{rr'}+\frac{1}{r'^2}, \text{ as in Art. 541.}$$

582. In order to see how a surface bends in different directions, starting from a given point, we ought to have a clear notion of the manner in which the normals at points adjacent to the given point are directed.

The indicatrix affords a satisfactory explanation of the mode of distribution.



583. *Normals at consecutive points intersect, when taken along the directions of greatest and least curvature, and not generally when taken in any other direction.*

Let P be any point of the indicatrix for the point O, CD a semi-diameter conjugate to CP, PT a tangent at P, PF perpendicular to CD.

Since the normal PS' to the surface at P is perpendicular to the tangent PT, PF is its projection on the plane of the indicatrix. Hence the normal PS' cannot intersect the normal at O unless CF vanish, which is the case only when P is at the extremities of the axes of the indicatrix; that is, when P is in one of the principal sections.

The same kind of argument shews that, in particular cases where the indicatrix is not a conic, the normal still intersects whenever the tangent at P is perpendicular to CP.

584. *When normals at consecutive points do not intersect, to find the direction and magnitude of the shortest distance between them.*

Let SS' be the shortest distance of the normals OS, PS', CF is its projection on the plane of the indicatrix, the shortest distance is therefore in the direction of the tangent conjugate to PT.

Also, $CF^2 = CP^2 - PF^2$ and $PF.CD = CA.CB$; therefore, if R, R' be the radii of curvature of the sections OP, OD, and ρ, ρ' those of the principal sections,

$$SS'^2 = CF^2 = CP^2\left(1 - \frac{\rho\rho'}{RR'}\right) \quad \text{(Art. 573)},$$

$$\text{where } R' = \rho + \rho' - R = \left(\frac{\rho'}{\rho}\cos^2\theta + \frac{\rho}{\rho'}\sin^2\theta\right) R.$$

585. *To find the point of nearest approach of a normal consecutive to a given normal.*

Draw $S'H$, FG perpendicular to the normal plane SOP, then PH is the projection of PS' on that plane; the tangent PU to the normal section is perpendicular to PS' and $S'H$, and therefore to PH, and R, the intersection of PH and OS, is ultimately the centre of curvature of the normal section OP.

Then $CS : CR :: PG : CP :: PF^2 : CP^2$;

$\therefore$ OS or $CS : R :: \rho\rho' : RR'$ ultimately;

hence OS is ultimately $= \dfrac{\rho\rho'}{R'}$, and $\dfrac{1}{OS} = R\left(\dfrac{\cos^2\theta}{\rho^2} + \dfrac{\sin^2\theta}{\rho'^2}\right)$.

586. *To find the angle between consecutive normals.*

The angle between the normals at O and P is ultimately

$$\frac{PF}{S'F} = \frac{PF}{CS} = \frac{PF}{R} \cdot \frac{CP}{PG} = \frac{CP}{R} \cdot \frac{CP}{PF} = \frac{CP}{R} \cdot \sqrt{\left(\frac{RR'}{\rho\rho'}\right)}$$

$$= CP\sqrt{\left(\frac{\cos^2\theta}{\rho^2} + \frac{\sin^2\theta}{\rho'^2}\right)}.$$

587. We leave to the student the calculation of the shortest distance and its position from the equation of the normal at the point whose coordinates are $r\cos\theta$, $r\sin\theta$, $\dfrac{r^2}{2R}$, viz.

$$\frac{\xi - r\cos\theta}{\dfrac{r\cos\theta}{\rho}} = \frac{\eta - r\sin\theta}{\dfrac{r\sin\theta}{\rho'}} = \frac{\zeta - \dfrac{r^2}{2R}}{-1}.$$

The expression for the shortest distance will be found to be

$$\frac{r\sin\theta\cos\theta\left(\dfrac{1}{\rho'} \sim \dfrac{1}{\rho}\right)}{\sqrt{\left(\dfrac{\cos^2\theta}{\rho^2} + \dfrac{\sin^2\theta}{\rho'^2}\right)}}.$$

588. *All the normals to a surface in the neighbourhood of a point converge to or diverge from two focal lines at right angles to one another.*

The equation of the surface being $\zeta = \dfrac{\xi^2}{2\rho} + \dfrac{\eta^2}{2\rho'} + \&c.$, the equations of a normal at $\left(r\cos\theta, r\sin\theta, \dfrac{r^2}{2R}\right)$ are

$$\frac{\rho(\xi - r\cos\theta)}{r\cos\theta} = \frac{\rho'(\eta - r\sin\theta)}{r\sin\theta} = \frac{\zeta}{-1} \text{ neglecting } r^2,$$

when $\eta = 0$, $\zeta = \rho'$, and when $\xi = 0$, $\zeta = \rho$, hence all normals in

the neighbourhood of O pass through two focal lines in the principal planes which pass respectively through the centres of curvature of the principal sections to which they are perpendicular. This theorem is due to Sturm.

589. Certain properties of the principal radii of curvature may be conveniently investigated by considering the angle between the two inflexional tangents. In these directions three consecutive points lie in a straight line, and the radius of curvature of a normal section through one of these tangents is therefore infinite. Hence, if θ be the angle which one of these tangents makes with the tangents to a section of principal curvature, we shall have $0 = \frac{\cos^2\theta}{\rho} + \frac{\sin^2\theta}{\rho'}$, ρ, ρ' being the algebraic magnitudes of the radii of principal curvature. Thus, for points at which the radii of principal curvature are equal in magnitude and opposite in sign, we shall have $\tan^2\theta = 1$, and the tangents to the curve of intersection will therefore also be at right angles. As an example of this method we shall take the following: To prove that at every point where the surface $x(x^2 + y^2 + z^2) = 2a(x^2 + y^2)$ meets the plane $x = a$, the radii of curvature will be equal in magnitude, and of opposite signs. This, by what has been said, will be true, if we can prove that the two straight lines, drawn through any such point, to meet the surface in three consecutive points, are at right angles to each other.

Let $\frac{\xi - a}{\lambda} = \frac{\eta - y}{\mu} = \frac{\zeta - z}{\nu} = r$ be a straight line which meets the surface in three consecutive points; the equation for determining r is

$$(a + \lambda r)(z + \nu r)^2 = (a - \lambda r)\{(a + \lambda r)^2 + (y + \mu r)^2\},$$

which must have its three roots equal to zero; the conditions of which are

$$z^2 = a^2 + y^2, \qquad (1)$$

$$y^2\lambda - ay\mu + az\nu = 0, \qquad (2)$$

$$2z\lambda\nu + a\nu^2 = -a\lambda^2 + a\mu^2 - 2y\lambda\mu, \qquad (3)$$

(3) becomes $z^2\lambda^2 + 2az\lambda\nu + a^2\nu^2 = y^2\lambda^2 - 2ay\lambda\mu + a^2\mu^2$,

$$\text{or } z\lambda + a\nu = \pm (y\lambda - a\mu) = \mp \frac{az}{y}\nu;$$

$$\therefore\ \lambda = \frac{a}{z}\left(-1 \mp \frac{z}{y}\right)\nu,$$

$$\mu = \left(\frac{z}{y} - \frac{y}{z} \mp 1\right)\nu = \left(\frac{a^2}{yz} \mp 1\right)\nu.$$

hence, if $(\lambda_1, \mu_1, \nu_1)$ $(\lambda_2, \mu_2, \nu_2)$ be the directions of the inflexional tangents,

$$\frac{\lambda_1\lambda_2}{\nu_1\nu_2} + \frac{\mu_1\mu_2}{\nu_1\nu_2} = \frac{a^2}{z^2}\left(1 - \frac{z^2}{y^2}\right) + \frac{a^4}{y^2z^2} - 1 = -1;$$

therefore $\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0$, or the two tangents are at right angles.

590. *The rectifying surface of a curve is the locus of the centres of principal curvature of the torse of which the curve is the edge of regression.*

The section of least curvature in a developable surface is that through a generating line, the normal section perpendicular to this line is therefore the section of greatest curvature. Now, in the figure of Art. 545, the plane uHt is ultimately the normal section of the torse perpendicular to PQq, and H is therefore the centre of principal curvature, every point of the axis of the osculating cone, *i.e.* of the rectifying line at P (Art. 546), is also such a centre, and the rectifying surface is the locus of all the centres of principal curvature of the developable whose edge of regression is the original curve.

Also the radius of principal curvature of a point in the developable, whose distance measured along a tangent to the curve is c, will therefore be $c \tan\psi = c\dfrac{d\varepsilon}{d\tau}$ (Art. 545).

591. In connexion with the curvature of surfaces, the most important lines which can be traced on a surface are *lines of curvature* and *geodesic lines*.

DEF. 1. A *Line of Curvature* is a curve traced upon a surface, such that the tangent to the curve at any point is also a tangent to one of the principal normal sections of the surface at that point.

Since there are two principal normal sections at every point, whose planes are at right angles, there will be two lines of curvature through every ordinary point, crossing one another at right angles.

DEF. 2. A *line of curvature* is a curve traced on a surface, such that the normals to the surface at any two consecutive points of the curve intersect each other.

That the curves given by these definitions are identical, is shewn in Art. 583.

592. DEF. A *geodesic line* of a given surface, between two given points on it, is a line of maximum or minimum length. Any infinitesimal arc of such a line will manifestly be the minimum line between its extremities, but if the two given points be at a finite distance, a geodesic passing through them may be either a maximum or minimum, and it will be seen that there may be an infinite number of such maxima and minima.

593. *The osculating plane at any point of a geodesic on any surface contains the normal to the surface.*

The distance between two indefinitely near points will be the least possible, when the curvature of the line joining them is the least, or when the radius of curvature is the greatest. Now the curvature of any curve on a surface will be, at any point, the same as that of the section of the surface made by the osculating plane at that point, since the two curves will have three coincident points. Also, of all sections having a common tangent line, the normal section is that of least curvature, by Meunier's Theorem (Art. 579). Hence, the osculating plane of a geodesic, at any point, must be a normal section, and the principal normal of the geodesic must coincide with the normal to the surface at every point.

This also appears from the consideration of a stretched weightless string, joining any two points on a smooth surface. This will manifestly assume the form of the shortest or longest line joining the points, the equilibrium being stable in the first case and unstable in the second, and since the resultant of the tensions of two consecutive elements of the string is balanced

by the normal reaction of the surface, the normal must lie in the plane of these elements, that is, in the osculating plane of the curve.

The coincidence of the principal normal and the normal to the surface $F(x, y, z) = 0$, on which a geodesic is drawn, is expressed by the equations

$$\frac{\dfrac{d^2x}{ds^2}}{\dfrac{dF}{dx}} = \frac{\dfrac{d^2y}{ds^2}}{\dfrac{dF}{dy}} = \frac{\dfrac{d^2z}{ds^2}}{\dfrac{dF}{dz}},$$

$$\text{and since } \frac{dF}{dx}\frac{dx}{ds} + \frac{dF}{dy}\frac{dy}{ds} + \frac{dF}{dz}\frac{dz}{ds} = 0,$$

$$\text{and } \frac{d^2x}{ds^2}\frac{dx}{ds} + \frac{d^2y}{ds^2}\frac{dy}{ds} + \frac{d^2z}{ds^2}\frac{dz}{ds} = 0,$$

these are equivalent to only one distinct equation which with that to the surface will give any geodesic line.

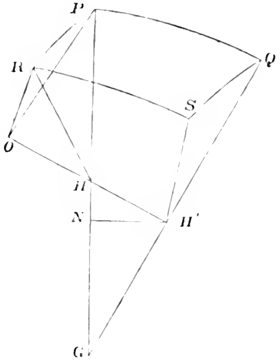
594. *Lines of curvature of a surface of revolution.*

Let PQ be an arc of the generating curve, PGO, QHO normals to the curve at P and Q intersecting the axis of revolution in G, H. When the plane of the curve turns round the axis GH, PQ comes into the position $P'Q'$, and the normals to the surface on P and P' intersect in G, also the normals at P and Q intersect in O; therefore the meridians and the circular sections are lines of curvature.

595. *To find the osculating plane of a line of curvature at any point of a surface.*

Let PQ, PR be small arcs of lines of curvature drawn through P, a point in the surface, RS, QS lines of curvature through R, Q respectively; and let PHG, $QH'G$, RH, SH' be normals to the surface at P, Q, R, S, so that PH, QH' are ultimately the radii of curvature of the principal normal sections PR, QS, and PG that of PQ; let these be R', $R'+dR'$, and R, dR' being the increment of R' due to a change ds along the principal section PQ.

The tangent to PR at P is perpendicular to the plane PHH', and therefore to HH', and the tangent at R is, for a similar reason, perpendicular to HH', which is therefore parallel to the



binormal at P to the line PR, and determines the osculating plane POR.

Let ϕ be the inclination of the osculating plane of PR to the principal normal section; draw $H'N$ perpendicular to PG,

$$\text{then } \tan\phi = \text{lim.} \frac{HN}{H'N} = \text{lim.} \frac{HN}{PQ} \cdot \frac{PQ}{H'N} = \frac{dR'}{ds} \cdot \frac{R}{R-R'} \cdot$$

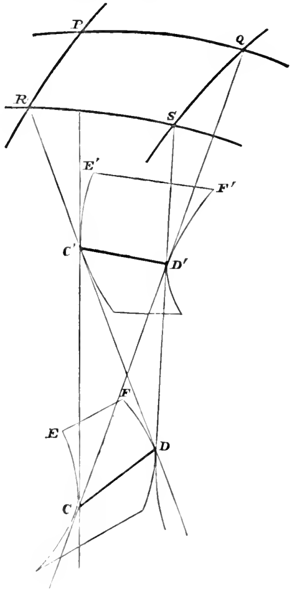
COR. In the case of a surface of revolution, since R' is the same for all points in the circular line of curvature supposed to correspond to R, $\frac{dR'}{ds} = 0$, and the osculating plane coincides with the normal plane.

Surface of Centres.

596. DEF. The surface of centres is the locus of the centres of principal curvature for every point of a surface.

597. Let L, L' be two lines of curvature passing through the point P of the surface S, and let Q, R be points on L and L' adjacent to P; then QC and RC', normals to S at Q and R,

intersect the normal at P in C, C' which are ultimately the centres of curvature of the normal sections touching L and L' respectively, and the planes CPQ, $C'PR$ are at right angles.



Normals drawn at every point of L form a torse whose edge of regression CE is the locus of the centres of curvature of all normal sections of S touching L, these normal planes being tangent planes to the torse.

If an infinite number of lines of curvature of the same system as L be traced on the surface, the corresponding edges such as CE, DF will form a sheet of the surface of centres, and a second sheet will be formed by edges $C'E'$, $D'F'$ corresponding to the lines of the system L'. Call these sheets Σ and Σ'.

Every normal of S such as $PC'C$, $SD'D$ touches each sheet; the two normals PC, QC each touch Σ', and since C does not lie on Σ', PCQ is a tangent plane to Σ', similarly $PC'R$ is a

tangent plane to Σ. And, since these two planes are at right angles, the two sheets would appear to cut one another at right angles to an eye situated in a normal and looking along the normal towards both sheets.

598. Salmon notices that the edge of regression of the torse corresponding to a line of curvature is a geodesic line on the sheet of the surface of centres on which it lies.

For PCQ is the osculating plane at C to the edge CE, and $PC'R$ is a tangent plane at C to Σ on which CE lies, and, since these planes are at right angles, the normal to Σ at C lies in the osculating plane of CE, which is the condition that CE should be a geodesic on Σ.

599. It may also be seen, that the polar developable of the line of curvature PQ is the rectifying developable of the corresponding edge of regression CE.

600. The two real sheets of the surface of centres for a surface of revolution are the surface generated by the evolute of the generating curve, and the portion of the axis from which normals can be drawn to the generating curve.

Lines of Curvature common to two Surfaces.

601. *When the curve of intersection of two surfaces is a line of curvature on each, the two surfaces cut one another at a constant angle.*

Let $PQRS$ in the figure of page 342 be ultimately the line of curvature common to two surfaces S, S'; and let pa, qa be normals to the surface S, which, since they intersect, must intersect in the polar line aUA, perpendicular to the osculating plane pUq; similarly qab, rb, normals at q and r, intersect in the polar line bV to the consecutive osculating plane, and

$$brV = bqV = aqU + UqV.$$

Let a', b' be corresponding points for the surface S';

$$\therefore\ b'rV = a'qU + UqV;$$

$$\therefore\ b'rb = a'qa;$$

hence, normals to SS' at consecutive points q, r are inclined at the same angle, therefore the surfaces cut one another throughout at a constant angle.

Reciprocally, *if two surfaces cut one another at a constant angle, and their curve of intersection be a line of curvature on one surface, it will be a line of curvature on the other also.*

Let it be a line of curvature on S, and let the normals to S' at q and r be $qa''b'$ and rb'' meeting Ua in a'' and Ub in b', b''; then since the angle between the normals to S and S' at q, r are equal, $brb'' = bqb'$, and $brb'' = bqb''$, therefore b' and b'' coincide, and the normals to S' at q, r intersect; that is, the curve is a line of curvature also on S'.

COR. If a line of curvature be a plane curve, its plane will cut the surface at a constant angle.

602. The analytical proof given by Bertrand is very simple. Let P, Q be consecutive points on the curve of intersection of surfaces S, S'; x, y, z and $x+dx$, $y+dy$, $z+dz$ their coordinates; l, m, n and l', m', n' the direction-cosines of the normals at P to S and S'.

If the curve be a line of curvature on S, the normals at P, Q will intersect;

$$\therefore\; x - l\rho = x + dx - (l+dl)\,\rho\,;$$

$$\therefore\; \frac{dl}{dx} = \frac{dm}{dy} = \frac{dn}{dz} = \frac{1}{\rho}. \tag{1}$$

Since PQ is perpendicular to both normals,

$$\begin{aligned} l\,dx + m\,dy + n\,dz &= 0,\\ \text{and}\quad l'dx + m'dy + n'dz &= 0. \end{aligned} \tag{2}$$

i. If the curve be a line of curvature on both surfaces,

$$\left.\begin{aligned} ldl' + mdm' + ndn' &= 0,\\ \text{and}\quad l'dl + m'dm + n'dn &= 0, \end{aligned}\right\}\ \text{by (1) and (2)},$$

$$\therefore\; d\,(ll' + mm' + nn') = 0,$$

or the cosine of the angle between the normals is constant.

ii. If the curve be a line of curvature on S, and the surfaces cut one another at a constant angle,

$$l'dl + m'dm + n'dn = 0,$$

$$\text{and } d(ll'+mm'+nn')=0;$$
$$\therefore\ ldl'+mdm'+ndn'=0,$$
$$\text{also } l'dl'+m'dm'+n'dn'=0;$$

therefore, by (2), $\dfrac{dl'}{dx}=\dfrac{dm'}{dy}=\dfrac{dn'}{dz}$, the condition that the curve should be a line of curvature on S'.

603. *When three series of surfaces cut one another orthogonally, the curve of intersection of any two of them is a line of curvature on each.*

Let the origin be a point of intersection of three surfaces, one of each series, and the tangents to their lines of intersection the axes. The equations of the three surfaces may then be written

$$x+ay^2+2byz+cz^2+\ldots=0, \tag{1}$$
$$y+a'z^2+2b'zx+c'x^2+\ldots=0, \tag{2}$$
$$z+a''x^2+2b''xy+c''y^2+\ldots=0. \tag{3}$$

At a consecutive point on the curve of intersection of (2) and (3), we have $y=0$, $z=0$, $x=x'$, and the equations of the tangent planes are, ultimately,

$$x.2c'x'+y+z.2b'x'=0,$$
$$x.2a''x'+y.2b''x'+z=0,$$

and since these also are at right angles,

$$4a''c'x'^2+2b''x'+2b'x'=0,$$

or, ultimately, $b'+b''=0$; similarly, $b''+b=0$, $b+b'=0$, which can only be satisfied by $b=0$, $b'=0$, $b''=0$, and therefore the axes are tangents to the lines of curvature on each surface.

Hence, the tangent lines, at any point of intersection of three surfaces, to their curves of intersection, are tangents to the lines of curvature of the three surfaces through that point, and, consequently, their curves of intersection must coincide with the lines of curvature. This is Dupin's theorem. A proof is given by Cayley,* which puts in evidence the geometrical ground on which the theorem rests.

* *Quarterly Journal*, vol. XII., p. 185.

Measure of Curvature.

604. Gauss gives the following definition of Integral and Total Curvature.

DEF. The *Integral Curvature* of any given portion of a curved surface is the area enclosed on a spherical surface of unit radius by a cone whose vertex is the centre, and whose generating lines are parallel to the normals to the surface at every point of the boundary of the given portion.

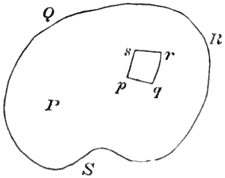
Horograph. The curve traced out on the sphere as described above is called the *horograph* of the given portion of the surface.

Average Curvature. The average curvature of any portion of a curved surface is the integral curvature divided by the area of the portion.

Specific Curvature. The specific curvature of a curved surface at any point is the average curvature of an infinitely small area including the point. This is the measure of curvature which was shewn by Gauss to be the reciprocal of the product of the two principal radii of curvature at the point considered.

605. *To shew that the reciprocal of the product of the principal radii at any point of a surface is a proper measure of the curvature.*

Let an elementary area QRS be described including the point P of a surface, and let a series of lines of curvature



divide this area into sub-elementary portions, such as $pqrs$, and let ρ_1, ρ_1' be the principal radii of curvature at p in the directions pq, ps; the horograph for $pqrs$ will be a small rectangle whose sides are $\dfrac{pq}{\rho_1}$ and $\dfrac{ps}{\rho_1'}$, and area $= \dfrac{pqrs}{\rho_1\rho_1'}$.

But, if ρ, ρ' be the principal radii of curvature at P,

$$\frac{1}{\rho_1\rho_1'} = \frac{1}{\rho\rho'}(1+\varepsilon),$$

where ε vanishes in the limit; therefore the specific curvature

$$= \lim. \frac{\Sigma \frac{pqrs}{\rho_1\rho_1'}}{\Sigma\, pqrs} = \frac{1}{\rho\rho'}.$$

This expression is independent of the form of the elementary portion including P, and is analogous to the measure of curvature in plane curves, the solid angle of the cone corresponding to the angle between the normals to a plane curve at the extremities of the small arc on which a point of the curve lies.

606. *To determine the radius of curvature of the normal section of a surface through a given tangent line at a given point in terms of the coordinates.*

Let the equation of the surface be $F(\xi, \eta, \zeta) = 0$; and let (x, y, z) be the given point P, (λ, μ, ν) the direction of the given tangent; also let $(x+dx, y+dy, z+dz)$ be a consecutive point Q taken in the normal section, so that ultimately

$$dx : dy : dz = \lambda : \mu : \nu.$$

Then, if QR be perpendicular to the tangent plane, R the radius of curvature of the normal section will be the limit of $\frac{PQ^2}{2QR}$.

The equation of the tangent plane is

$$U(\xi - x) + V(\eta - y) + W(\zeta - z) = 0;$$

$$\therefore\ QR = \frac{Udx + Vdy + Wdz}{\pm P},$$

$$\text{where } P^2 = U^2 + V^2 + W^2.$$

But, Q being a point on the surface,

$$Udx + Vdy + Wdz + \tfrac{1}{2}\{u\,(dx)^2 + \ldots + 2u'dydz + \ldots\} = 0,$$

neglecting terms of degrees higher than the second;

$$\therefore\ R = \pm \frac{\{(dx)^2 + (dy)^2 + (dz)^2\}P}{2\,(Udx + Vdy + Wdz)}$$

$$= \mp \frac{P}{u\lambda^2 + v\mu^2 + w\nu^2 + 2u'\mu\nu + 2v'\nu\lambda + 2w'\lambda\mu}.$$

Since we have the conditions

$$U\lambda + V\mu + W\nu = 0 \text{ and } \lambda^2 + \mu^2 + \nu^2 = 1,$$

the problem of finding the directions of the principal sections and the magnitude of the principal radii of curvature is the same as that of finding the direction and magnitude of the principal axes of the section of the conicoid,

$$ux^2 + \ldots + 2u'yz + \ldots = 1,$$

made by the plane $Ux + Vy + Wz = 0$.

607. *To determine the principal normal sections, and the radii of principal curvature at any point of a surface, in terms of the coordinates of the point.*

The radius of a normal section containing the tangent whose direction is (λ, μ, ν) is given by

$$u\lambda^2 + v\mu^2 + w\nu^2 + 2u'\mu\nu + 2v'\nu\lambda + 2w'\lambda\mu - \frac{P}{R}(\lambda^2 + \mu^2 + \nu^2) = 0, \quad (1)$$

$$\text{where } U\lambda + V\mu + W\nu = 0, \quad (2)$$

and when R is given, the corresponding tangent lines are the lines of intersection of the cone and plane represented by these equations, λ, μ, ν being considered current coordinates. When R is a maximum or minimum, these directions coincide, and the plane is a tangent plane of the cone; hence the direction of the principal sections are given by

$$\frac{(u-\sigma)\lambda + w'\mu + v'\nu}{U} = \frac{w'\lambda + (v-\sigma)\mu + u'\nu}{V}$$

$$= \frac{v'\lambda + u'\mu + (w-\sigma)\nu}{W}, \text{ where } \sigma = \frac{P}{R},$$

whence we obtain

$$\frac{\lambda}{U\{(v-\sigma)(w-\sigma) - u'^2\} + V\{u'v' - w'(w-\sigma)\} + W\{w'u' - v'(v-\sigma)\}}$$

$$= \frac{\mu}{\ldots\ldots} = \frac{\nu}{\ldots\ldots},$$

which, by the equation $U\lambda + V\mu + W\nu = 0$, leads to

$$U^2\{(v-\sigma)(w-\sigma) - u'^2\} + \ldots + 2VW\{v'w' - u'(u-\sigma)\} + \ldots = 0. \quad (3)$$

This equation gives the values of the principal radii of curvature, and the values of $\lambda : \mu : \nu$, corresponding to each root, are given by the preceding system of equations.

COR. The product of the roots of (3) is

$$\frac{U^2(vw - u'^2) + \ldots + 2VW(v'w' - uu') + \ldots}{U^2 + V^2 + W^2}$$

$$= (U^2 + V^2 + W^2) \times \text{measure of curvature.}$$

Since the measure of curvature vanishes at every point of a developable surface, the numerator equated to zero is the condition that a surface should be developable.

608. We cannot help calling attention to another form of the quadratic giving the principal radii, which was set in an examination paper for Clare and Caius Colleges in 1873.

Since $2VW\mu\nu = U^2\lambda^2 - V^2\mu^2 - W^2\nu^2$, &c., the expression for $\dfrac{P}{R}$ can be put into the form $A\lambda^2 + B\mu^2 + C\nu^2$, where

$$A = u + \frac{U}{VW}(Uu' - Vv' - Ww'), \text{ \&c.}$$

Construct the conicoid $A\xi^2 + B\eta^2 + C\zeta^2 = P$, having its centre at the point (x, y, z) of the surface, the directions of the axes of the section made by the plane $U\xi + V\eta + W\zeta = 0$ are the directions of the tangents to the principal sections of the surface, and the corresponding values of R will be the squares of the section.

Hence, by Art. 234,

$$\frac{U^2}{AR-P} + \frac{V^2}{BR-P} + \frac{W^2}{CR-P} = 0,$$

a quadratic giving ρ, ρ' the principal radii of curvature.

Also the direction cosines of the tangents to the lines of curvature are as $\dfrac{U}{AR-P} : \dfrac{V}{BR-P} : \dfrac{W}{CR-P}$, where ρ_1, ρ_2 are to be written for R.

609. *To determine the conditions of an umbilic.*

At an umbilic R retains a constant value for all directions (λ, μ, ν), satisfying the two conditions (1) and (2). Hence at an umbilic the cone (1) must break up into two planes, one of which is the tangent plane (2).

The left-hand member of equation (2) must therefore be a factor of the left-hand member of (1), and the other factor will therefore be

$$\frac{\lambda}{U}(u-\sigma)+\frac{\mu}{V}(v-\sigma)+\frac{\nu}{W}(w-\sigma).$$

Multiplying the two, and equating coefficients,

$$\frac{V}{W}(w-\sigma)+\frac{W}{V}(v-\sigma)=2u',$$

$$\frac{W}{U}(u-\sigma)+\frac{U}{W}(w-\sigma)=2v',$$

$$\frac{U}{V}(v-\sigma)+\frac{V}{U}(u-\sigma)=2w',$$

which, on eliminating σ, lead to the two conditions

$$\frac{W^2v+V^2w-2VWu'}{V^2+W^2}=\frac{U^2w+W^2u-2WUv'}{W^2+U^2}$$
$$=\frac{V^2u+U^2v-2UVw'}{U^2+V^2}.$$

These two equations, together with the equation of the surface, will, in general, determine a definite number of points, among which are included all the umbilici. It may happen that a common factor exists, so that the three equations are satisfied by the coordinates of any point lying on a certain curve. Such a curve is called *a line of spherical curvature.*

It should also be observed that U, V, W have been assumed to be *finite* in the foregoing investigation. Should one of them, as U, vanish, we must have, in the same manner, $V\mu+W\nu$ a factor, and must therefore have

$$(u-\sigma)\lambda^2+\ldots+2u'\mu\nu+\ldots$$
$$\equiv(V\mu+W\nu)\left\{k\lambda+(v-\sigma)\frac{\mu}{V}+(w-\sigma)\frac{\nu}{W}\right\}.$$

This identity gives

$$u=\sigma,\ \ 2u' = (v-\sigma)\frac{W}{V} + (w-\sigma)\frac{V}{W},\ \ 2v' = kW,\ \ 2w' = kV,$$

$$\text{or}\ \ Vv' = Ww',\ \ 2u' = (v-u)\frac{W}{V} + (w-u)\frac{V}{W},$$

which with $U=0$, and the equation of the surface, give *four* relations between the coordinates, unless v' and w' are identically zero, and these will not, in general, be simultaneously true of any point on the surface.

610. The conditions for an umbilic are obtained by the method of Art. 608, from the consideration that the section of the conicoid must be circular, whence, when U, V, W are finite, it follows that $A=B=C$.

611. *To determine the number of umbilics on a surface of the* n^{th} *degree.*

If the equations for an umbilic be written in the form

$$\frac{P}{P'} = \frac{Q}{Q'} = \frac{R}{R'},$$

the degree of P', Q', R' will be $2(n-1)$, and of P, Q, R will be $3n-4$. The degree of the surfaces

$$QR' - Q'R = 0,\ RP' - R'P = 0$$

is therefore $5n-6$, and the degree of their curve of intersection is $(5n-6)^2$. But the curve $R=0$, $R'=0$ is part of their intersection, and does not lie on the surface $PQ' - P'Q = 0$. The degree of the curve

$$\frac{P}{P'} = \frac{Q}{Q'} = \frac{R}{R'}$$

is therefore $(5n-6)^2 - 2(n-1)(3n-4) = 19n^2 - 46n + 28$.

But this curve includes three curves similar to

$$U=0,\ \ u = \frac{W^2v + V^2w - 2VWu'}{V^2+W^2},$$

which do not meet the surface in umbilics, and the degree of this curve is $(n-1)(3n-4)$.

Hence the degree of the curve, which meets the surface in umbilics only, is

$$19n^2 - 46n + 28 - 3(n-1)(3n-4) \equiv 10n^2 - 25n + 16.$$

The whole number of real and impossible umbilics is therefore

$$n(10n^2 - 25n + 16).$$

Thus in a conicoid the number is 12, four in each of the principal planes; but not more than one system is real, and, if the surface be a ruled surface, none will be real.

There can never be real umbilics on a ruled surface of any degree whatever, since every point of a ruled surface is either parabolic or hyperbolic.

612. *To find the differential equations of the lines of curvature on any surface.*

Referring to the equations of Art. 607, we see that λ, μ, ν are the direction-cosines of the tangents to the planes of principal curvature at any point, and are therefore the direction-cosines of the tangents to the lines of curvature through that point.

Hence, if (x, y, z) be the point, $(x+dx,\ y+dy,\ z+dz)$ a consecutive point on a line of curvature, we shall have

$$\frac{dx}{\lambda} = \frac{dy}{\mu} = \frac{dz}{\nu},$$

which, since λ, μ, ν are determined in terms of x, y, z, are the differential equations of the lines of curvature.

Since each member of the above equations is equal to

$$\frac{Udx + Vdy + Wdz}{U\lambda + V\mu + W\nu} = \frac{0}{0},$$

these equations and the equation of the surface are equivalent only to two independent equations.

Any integral we may find will involve one arbitrary constant, which may be determined so as to make the line of curvature pass through any proposed point on the surface.

It appears from the above that all the surfaces represented by the equation $\phi(x, y, z) = c$, for different values of c, will have the same differential equations of their lines of curvature.

613. *To find the differential equation of the lines of curvature, and the principal radii of curvature at any point, by the second definition.*

Let (ξ, η, ζ) be the point of intersection of normals at consecutive points (x, y, z) and $(x+dx, y+dy, z+dz)$;

$$\therefore \frac{\xi - x}{U} = \frac{\eta - y}{V} = \frac{\zeta - z}{W} = \frac{\rho}{P} = \frac{1}{\sigma}, \text{ where } P^2 = U^2 + V^2 + W^2,$$

in which equations ξ, η, ζ are unaltered when $x + dx$ is written for x, &c., and $\xi = x + U\dfrac{\rho}{P}$, &c.,

$$\therefore \ 0 = dx + \frac{1}{\sigma}\, dU - U\frac{1}{\sigma^2}\, d\sigma,$$

$$0 = dy + \frac{1}{\sigma}\, dV - V\frac{1}{\sigma^2}\, d\sigma,$$

$$0 = dz + \frac{1}{\sigma}\, dW - W\frac{1}{\sigma^2}\, d\sigma;$$

$$\therefore \begin{vmatrix} dx, & dU, & U \\ dy, & dV, & V \\ dz, & dW, & W \end{vmatrix} = 0,$$

which is the differential equation of the lines of curvature.

Expanding dU, dV, dW, and eliminating dx, dy, dz, and $d\sigma$,

$$\begin{vmatrix} u-\sigma, & w', & v', & U \\ w', & v-\sigma, & u', & V \\ v', & u', & w-\sigma, & W \\ U, & V, & W & \end{vmatrix} = 0.$$

The coefficient of U^2 is $-\begin{vmatrix} v-\sigma, & u' \\ u', & w-\sigma \end{vmatrix}$, and that of VW is $\begin{vmatrix} u-\sigma, & v' \\ w', & u' \end{vmatrix}$, whence we obtain the quadratic given in Art. 607.

614. The foregoing equations for determining the principal curvatures undergo a considerable simplification, if the equation of the surface be of the form

$$\phi_1(x) + \phi_2(y) + \phi_3(z) = 0.$$

We shall then have u', v', w' all zero; the equation giving the length of the radius of curvature of any normal section, whose tangent line is (λ, μ, ν), will be

$$\frac{P}{R} = u\lambda^2 + v\mu^2 + w\nu^2;$$

the quadratic equation for the principal radii of curvature will be

$$\frac{U^2}{Ru - P} + \frac{V^2}{Rv - P} + \frac{W^2}{Rw - P} = 0;$$

the differential equation of the lines of curvature will be

$$U(v - w)\,dy\,dz + V(w - u)\,dz\,dx + W(u - v)\,dx\,dy = 0.$$

The conditions for an umbilic in this case reduce to $u = v = w$ when U, V, W are finite, but since this is the exceptional case mentioned in Art. 609, in which u', v', and w' vanish identically, there are other umbilics which are given by $U = 0$ and $(v - u)W^2 + (w - u)V^2 = 0$, and similar equations when $V = 0$ and $W = 0$. The whole number of umbilics is therefore, as before,

$$n\{(n-2)^2 + 3(n-1)(3n-4)\} \equiv n(10n^2 - 25n + 16).$$

615. *To obtain the differential equation of the lines of curvature, and to find the centres and radii of principal curvature when the equation of the surface gives one of the coordinates explicitly in terms of the other two.*

Let the equation of the surface be $\zeta = f(\xi, \eta)$, and let P, Q be consecutive points on a line of curvature whose coordinates are x, y, z, and $x + dx$, $y + dy$, $z + dz$, then the normals at P, Q intersect; and if (ξ, η, ζ) be their point of intersection,

$$\xi - x + p(\zeta - z) = 0, \text{ and } \eta - y + q(\zeta - z) = 0, \qquad (1)$$

but ξ, η, ζ remain the same when $x + dx$, $y + dy$, $z + dz$ are written for x, y, z; therefore

$$dp(\zeta - z) = dx + p\,dz, \text{ and } dq(\zeta - z) = dy + q\,dz; \qquad (2)$$

$$\therefore \frac{r\,dx + s\,dy}{s\,dx + t\,dy} = \frac{dx + p(p\,dx + q\,dy)}{dy + q(p\,dx + q\,dy)};$$

$$\therefore \{(1 + q^2)s - pqt\}(dy)^2 + \{(1 + q^2)r - (1 + p^2)t\}\,dx\,dy$$
$$- \{(1 + p^2)s - pqr\}(dx)^2 = 0, \qquad (3)$$

which is the differential equation of the projection of the lines of curvature on the plane of xy.

Let ρ be the radius of curvature of the principal section through PQ, hence by (1) $\rho^2 = (1+p^2+q^2)(z-\zeta)^2$, therefore, writing in (2) σ for $z-\zeta$ or $\dfrac{\rho}{\sqrt{(1+p^2+q^2)}}$,

$$(rdx + sdy)\,\sigma + dx + p\,(pdx + qdy) = 0,$$

$$\therefore\ (r\sigma + 1 + p^2)\,dx + (s\sigma + pq)\,dy = 0,$$

$$\text{and similarly}\ \ (t\sigma + 1 + q^2)\,dy + (s\sigma + pq)\,dx = 0;$$

$$\therefore\ (r\sigma + 1 + p^2)(t\sigma + 1 + q^2) - (s\sigma + pq)^2 = 0,$$

or $(rt - s^2)\,\sigma^2 + \{(1+q^2)\,r - 2pqs + (1+p^2)\,t\}\,\sigma$

$$+ 1 + p^2 + q^2 = 0. \qquad (4)$$

COR. Gauss' measure of curvature is

$$\frac{1}{\rho\rho'} = \frac{1}{\sigma\sigma'}\,\frac{1}{1+p^2+q^2} = \frac{rt-s^2}{(1+p^2+q^2)^2} \text{ by (4)},$$

which vanishes for a developable surface.

616. *To find the umbilics of the surface $z = f(x, y)$.*

Since the normals at points passing in any direction from an umbilic intersect the normal at the umbilic, neglecting small quantities of the third order in dx and dy, the equation (3) must be true independently of the value of $dy : dx$, and this condition is satisfied by

$$\frac{1+p^2}{r} = \frac{pq}{s} = \frac{1+q^2}{t};$$

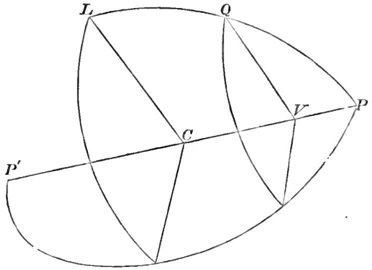
these equations, with the equation of the surface, determine the umbilics.

Curvature of Conicoids.

617. *To find the radii of principal curvature at any point of a central conicoid.*

Let P be any point on the conicoid, supposed in the figure to be an ellipsoid, PCP' the diameter through P, CL the radius parallel to the tangent at P to any normal section whose radius of curvature is required, PQL the central section having the

same tangent. Let a plane be drawn through a point Q near P parallel to the tangent plane at P, meeting CP in V, and



let p, ϖ be the perpendiculars on the tangent plane from C and Q, so that $\varpi : p :: VP : CP$. The radius of curvature of the normal section is the limit of $\frac{PQ^2}{2\varpi}$ or $\frac{QV^2}{2\varpi}$, and

$$QV^2 : CL^2 :: PV.VP' : CP^2$$
$$:: \varpi.VP' : p.CP = 2\varpi : p \text{ ultimately,}$$

hence the radius of curvature $= \frac{CL^2}{p}$.

If α, β be the semi-axes of the central section parallel to the tangent plane, $\frac{\alpha^2}{p}$ and $\frac{\beta^2}{p}$ will be the principal radii of curvature, which we shall call ρ and ρ'.

618. *To find the coordinates of the centres of curvature.*

Let the equation of the conicoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad (1)$$

and let (f, g, h) be the point P, then ξ, η, ζ, the coordinates of the centre of curvature corresponding to ρ, satisfy the equations

$$\frac{\xi - f}{\frac{f}{a^2}} = \frac{\eta - g}{\frac{g}{b^2}} = \frac{\zeta - h}{\frac{h}{c^2}} = -p\rho = -\alpha^2;$$

$$\therefore\ \xi = f\left(1 - \frac{\alpha^2}{a^2}\right),\ \ \eta = g\left(1 - \frac{\alpha^2}{b^2}\right),\ \ \zeta = h\left(1 - \frac{\alpha^2}{c^2}\right),$$

and by Art. 285, if the equations of confocal conicoids through P be

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1 \text{ and } \frac{x^2}{a''^2}+\frac{y^2}{b''^2}+\frac{z^2}{c''^2}=1,$$

α^2 and β^2 are respectively $a^2-a''^2$ and $a^2-a'^2$; therefore the coordinates of the centres of curvature are

$$\frac{fa''^2}{a^2},\ \frac{gb''^2}{b^2},\ \frac{hc''^2}{c^2} \text{ and } \frac{fa'^2}{a^2},\ \frac{gb'^2}{b^2},\ \frac{hc'^2}{c^2}.$$

619. *If three confocal conicoids (A), (B), (C) intersect in P, the centres of principal curvature of (A) at P are the poles with respect to (B) and (C) of the tangent plane to (A) at P.*

Let the three conicoids (A), (B), and (C) be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ \frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1, \text{ and } \frac{x^2}{a''^2}+\frac{y^2}{b''^2}+\frac{z^2}{c''^2}=1,$$

intersecting in (f, g, h).

The coordinates of the centre of curvature of the normal section containing the tangent to the intersection of (A) and (B) are $\frac{fa''^2}{a^2}$, $\frac{gb''^2}{b^2}$, $\frac{hc''^2}{c^2}$, and its polar, with respect to (C), is $\frac{fx}{a^2}+\frac{gy}{b^2}+\frac{hz}{c^2}=1$, the tangent plane to (A) at P.

Similarly for the other centre of principal curvature. This proposition is due to Salmon.

620. *The curve of intersection of two confocal conicoids is a line of curvature on each.*

Let PT be a tangent at P to the curve of intersection of two confocals S and S', PN, PN' normals at P to S and S'; and suppose a central section of S made by a plane parallel to the tangent plane $N'PT$, and therefore to the indicatrix to S at P. Now it is shewn (Art. 285) that PN' is parallel to one axis of this section; therefore PT is parallel to the other axis; hence, the tangent to the curve of intersection of S and S' at any point is parallel to an axis of the indicatrix of either surface at that point, and the curve is a line of curvature.

621. *At any point in a line of curvature of a conicoid, the rectangle contained by the diameter parallel to the tangent at that point and the perpendicular from the centre on the tangent plane at the point is constant.*

Let the line of curvature on the conicoid S be the curve of intersection with S', and let PT be a tangent to it at any point P; PN, PN' normals to S and S' at P; then, if α, β be the semi-axes of the central section parallel to $N'PT$, the tangent plane to S, which are parallel respectively to PT and PN', it is shewn (Cor., Art. 285) that β is constant, and if p be the perpendicular from the centre on the tangent plane, $p\alpha\beta$ is constant, therefore $p\alpha$ is constant.

622. The following proof is independent of the properties of confocal surfaces.

Let P, Q be consecutive points on a line of curvature, O the corresponding centre of curvature, ρ the radius of curvature, p the perpendicular on the tangent plane, α, β the semiaxes of the section parallel to the tangent plane, α being parallel to PQ, and let $CP = r$, then by the triangle OCP

$$OC^2 = \rho^2 + r^2 - 2\rho p,$$

and since, for a change from P to Q, O, C are unchanged in position and ρ is unaltered, $rdr = \rho dp$. But, by Art. 273,

$$\alpha^2 + \beta^2 + r^2 = a^2 + b^2 + c^2, \quad \alpha\beta p = abc;$$

$$\therefore \ \alpha d\alpha + \beta d\beta + rdr = 0, \ \text{ and } \ \frac{d\alpha}{\alpha} + \frac{d\beta}{\beta} + \frac{dp}{p} = 0,$$

multiplying the last equation by α^2 or $p\rho$, and subtracting the preceding, we obtain $(\alpha^2 - \beta^2)\, d\beta = 0$; therefore β is constant, unless $\alpha = \beta$, which is only true at an umbilic, therefore $p\alpha$ is also constant.

623. *To shew that the curves of intersection of a given conicoid with all confocal conicoids which intersect it satisfy the differential equations of a line of curvature.*

Let the equation of the surface be

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1; \qquad (1)$$

then the differential equations of the lines of curvature are

$$\frac{x}{a}\,dx + \frac{y}{b}\,dy + \frac{z}{c}\,dz = 0, \qquad (2)$$

$$x(b-c)\,dydz + y(c-a)\,dzdx + z(a-b)\,dxdy = 0. \qquad \text{(Art. 614)}$$

But for the curve of intersection of (1) with a confocal surface,

$$\frac{x^2}{a+k} + \frac{y^2}{b+k} + \frac{z^2}{c+k} = 1, \qquad (3)$$

we have $$\frac{xdx}{a(b-c)(a+k)} = \frac{ydy}{b(c-a)(b+k)} = \frac{zdz}{c(a-b)(c+k)}$$

$$= \frac{\dfrac{x^2}{a(a+k)} + \dfrac{y^2}{b(b+k)} + \dfrac{z^2}{c(c+k)}}{\dfrac{x(b-c)}{dx} + \dfrac{y(c-a)}{dy} + \dfrac{z(a-b)}{dz}},$$

and subtracting (3) from (1), we have

$$\frac{x^2}{a(a+k)} + \frac{y^2}{b(b+k)} + \frac{z^2}{c(c+k)} = 0$$

at all points of the curve of intersection. Hence we must have, at all points on such a curve,

$$\frac{x(b-c)}{dx} + \frac{y(c-a)}{dy} + \frac{z(a-b)}{dz} = 0,$$

that is, the differential equations of the curve of intersection of the given conicoid with any confocal surface coincides with the differential equations of the lines of curvature; and the equations of such curves, involving an arbitrary constant k, are therefore the complete integral of the differential equations of the lines of curvature.

Having given any one point (x', y', z'), we shall have the quadratic equation

$$\frac{x'^2}{a(a+k)} + \frac{y'^2}{b(b+k)} + \frac{z'^2}{c(c+k)} = 0$$

for determining k, and to each value we shall have a corresponding line of curvature passing through the point (x', y', z').

624. *To find the lines of curvature of a central conicoid from the differential equation of their projections on a principal plane.*

The differential equation of the projection of the lines of curvature on the plane of xy may be obtained either by eliminating dz from the equations (2) of the last article, or by Art. 615; the equation is

$$(b-c)\frac{xy}{b}(dy)^2+\left\{(a-c)\frac{x^2}{a}+(a-b)\frac{y^2}{b}+(b-a)\right\}dxdy$$
$$+(c-a)\frac{xy}{a}(dx)^2=0. \qquad (1)$$

Multiply by $\frac{4xy}{ab}$, and assume $\frac{x^2}{a}=u$, $\frac{y^2}{b}=v$; therefore

$$(b-c)u(dv)^2+\{(a-c)u+(c-b)v+b-a\}\,dudv+(c-a)v(du)^2=0,$$

$$\text{or } \{(b-c)dv-(c-a)du\}(udv-vdu)+(b-a)dudv=0. \qquad (2)$$

If we assume $v=\alpha+\alpha_1u+\ldots+\alpha_ru^r+\ldots$,

$$udv-vdu=(-\alpha+\alpha_2u^2+2\alpha_3u^3+\ldots)\,du,$$

hence the equation (2) cannot be identically satisfied unless α_2, $\alpha_3\ldots$ are all zero, and substituting $\alpha+\alpha_1u$ for v,

$$\{(b-c)\alpha_1-(c-a)\}(-\alpha)+(b-a)\alpha_1=0. \qquad (3)$$

The solution $v=\alpha+\alpha_1u$ is therefore the complete solution, since it involves one arbitrary constant in the second degree.

The projections of the lines of curvature are therefore conics of the form $\frac{x^2}{a'}+\frac{y^2}{b'}=1$, where $b'=b\alpha$, $a'=\frac{-a\alpha}{\alpha_1}$, so that, dividing (3) by α_1,

$$(a-c)\frac{a'}{a}+(c-b)\frac{b'}{b}+b-a=0. \qquad (4)$$

It can be shewn from this relation between the axes, or directly from the singular solution of the differential equation (1), that the system of conics is enveloped by the four straight lines

$$x\sqrt{\frac{a-c}{a}}\pm y\sqrt{\frac{c-b}{b}}=\pm\sqrt{(a-b)};$$

also, that each of these four straight lines is the projection of a generating line containing three umbilics real or imaginary.

The projections of the intersection of the two confocals (1) and (3) of Art. 623 are $\dfrac{(a-c)\,x^2}{a\,(a+k)} + \dfrac{(b-c)\,y^2}{b\,(b+k)} = 1$, the axes of which satisfy the condition (4), and the solutions agree.

625. *Lines of curvature of the paraboloids.*

Let $2z = \dfrac{x^2}{a} + \dfrac{y^2}{b}$ (1) be the equation of a paraboloid, the differential equation of the projections of the lines of curvature on the plane of xy is, by Art. 615,

$$\frac{xy}{ab^2}(dy)^2 + \left(\frac{1}{b} - \frac{1}{a} + \frac{x^2}{a^2b} - \frac{y^2}{ab^2}\right) dx\,dy - \frac{xy}{a^2b}(dx)^2 = 0,$$

the solution of which may be obtained as in Art. 621, viz. $\dfrac{x^2}{a'} + \dfrac{y^2}{b'} = 1$ (2), where a', b' are connected by the equation

$$\frac{a'}{a} - \frac{b'}{b} + a - b = 0.$$

The equation of a paraboloid, confocal with (1), is

$$2z + k = \frac{x^2}{a+k} + \frac{y^2}{b+k},$$

and of the projection of the curve of intersection

$$\frac{x^2}{a\,(a+k)} + \frac{y^2}{b\,(b+k)} + 1 = 0,$$

which is one of the system of conics (2).

626. The differential equation of the lines of curvature of a hyperbolic paraboloid, whose equation is $az = xy$, is

$$(a^2 + x^2)\,(dy)^2 - (a^2 + y^2)\,(dx)^2 = 0,$$

and the lines of curvature are the intersections of the paraboloid and hyperbolic cylinders, whose equations are

$$y^2 - 2Cxy + x^2 = a^2\,(C^2 - 1),$$

the positive and negative values of C determining the two systems of lines.

Lines of Curvature through an Umbilic.

627. *To shew that there are three directions passing from an umbilic in which the normals at the consecutive points intersect.*

If the axis of z be a normal at an umbilic, the equation of the surface is of the form $\zeta = \alpha(\xi^2 + \eta^2) + u_3(1 + \varepsilon)$, where u_3 is of the third degree in ξ and η, and ε vanishes in the limit; the equations of a normal at (x, y, z) are

$$\xi - x + \left(2\alpha x + \frac{du_3}{dx}\right)(\zeta - z) = 0,$$

$$\eta - y + \left(2\alpha y + \frac{du_3}{dy}\right)(\zeta - z) = 0;$$

but if this normal meet that at the umbilic, the equations are satisfied by $\xi = 0$, $\eta = 0$;

$$\therefore\ x\frac{du_3}{dy} - y\frac{du_3}{dx} = 0,$$

which gives three directions in which the point (x, y, z) must be taken.

628. *To find the three directions for which normals to a conicoid intersect the normal at an umbilic.*

Let $a\xi^2 + b\eta^2 + c\zeta^2 = 1$ (1) be the conicoid, $(\alpha, 0, \gamma)$ the umbilic, $(\alpha + \lambda r, \mu r, \gamma + \nu r)$ a point adjacent to it in the direction (λ, μ, ν), the equations of the two normals are

$$\frac{\xi - \alpha - \lambda r}{a(\alpha + \lambda r)} = \frac{\eta - \mu r}{b\mu r} = \frac{\zeta - \gamma - \nu r}{c(\gamma + \nu r)},$$

$$\text{and } \frac{\xi - \alpha}{a\alpha} = \frac{\zeta - \gamma}{c\gamma},\ \eta = 0;$$

one condition that they may intersect is $\mu = 0$, one direction is therefore that of the principal section containing the umbilic; for the other conditions

$$\xi - \alpha = \lambda r - \frac{a}{b}(\alpha + \lambda r) \text{ and } \zeta - \gamma = \nu r - \frac{c}{b}(\gamma + \nu r);$$

$$\therefore\ c\gamma(b - a)\lambda = a\alpha(b - c)\nu;$$

$$\text{or, since } \frac{a^2\alpha^2}{b - a} = \frac{c^2\gamma^2}{c - b},$$

$$a\alpha\lambda + c\gamma\nu = 0\,; \qquad (2)$$

and, by (1), $a(\alpha+\lambda r)^2 + b\mu^2 r^2 + c(\gamma+\nu r)^2 = 1\,;$ $\quad(3)$

$$\therefore\ a\lambda^2 + b\mu^2 + c\nu^2 = 0\,; \qquad (4)$$

(2) and (4) give the two other directions for which the normals intersect; and, since (3) is satisfied for all values of r, they are the directions of the imaginary generatrices through the umbilic.

629. We may observe also that since

$$a^2\alpha^2\lambda^2 = c^2\gamma^2\nu^2,\ (b-a)\lambda^2 = (c-b)^2\,;$$

$$\therefore\ a\lambda^2 + b\mu^2 + c\nu^2 = b\,(\lambda^2+\mu^2+\nu^2)\,;$$

$$\therefore\ \lambda^2+\mu^2+\nu^2 = 0 \text{ by (4)},$$

which shews that these generatrices pass through the imaginary circle at infinity.

Since the argument of Art. 628 is independent of the magnitude of r, it is true that all the normals at points along one of the umbilic at generatrices intersect, and they have therefore this character of lines of curvature, but Cayley has remarked in a note on a paper upon "the direction of lines of curvature in the neighbourhood of an umbilicus,"* that they are the envelopes of the lines of curvature, and belong to the singular solution of the differential equation of these lines, as appears from Art. 624.

630. In the note referred to above, Cayley remarks that since, at an umbilic, $\dfrac{dy}{dx}$ is determined by a cubic equation, there are generally three directions of the line of curvature, which may arise from three distinct curves, or from a curve with a triple point at the umbilic; and, referring to a paper by Serret,† he states that the lines of curvature on the surface $xyz = 1$ are its intersection with the series of surfaces

$$h = (x^2+\omega y^2+\omega^2 z^2)^{\frac{3}{2}} + (x^2+\omega^2 y^2+\omega z^2)^{\frac{3}{2}},$$

* Frost, *Quart. Journ. of Math.*, vol. x., p. 78, and Cayley, *ibid.* p. 111.

† *Liouv. Journ.*, t. 12 (1847), pp. 241—254.

where ω is an imaginary cube root of unity; now at the umbilic (1, 1, 1), corresponding to which $h = 0$,

$$(x^2 + \omega y^2 + \omega^2 z^2)^3 = (x^2 + \omega^2 y^2 + \omega z^2)^3;$$

$$\therefore\ x^2 + \omega y^2 + \omega^2 z^2 = x^2 + \omega^2 y^2 + \omega z^2,$$

$$\text{or } = \omega(x^2 + \omega^2 y^2 + \omega z^2), \text{ or } = \omega^2(x^2 + \omega^2 y^2 + \omega z^2);$$

$$\therefore\ y^2 = z^2, \text{ or } x^2 = y^2, \text{ or } z^2 = x^2;$$

hence, through the umbilic (1, 1, 1) three distinct lines of curvature pass, viz. the curves

$$y = z,\ xy^2 = 1;\quad x = y,\ zx^2 = 1;\ \text{and}\ z = x,\ yz^2 = 1.$$

631. The differential equation of the line of curvature of $xyz = 1$ is

$$xdydz(y^2 - z^2) + ydzdx(z^2 - x^2) + zdxdy(x^2 - y^2) = 0.$$

Multiply by xyz, and let $x^2 = p$, $y^2 = q$, $z^2 = r$;

$$\therefore\ p(q - r)dqdr + q(r - p)drdp + r(p - q)dpdq = 0. \quad (1)$$

Again, if $h = (p + \omega q + \omega^2 r)^{\frac{3}{2}} + (p + \omega^2 q + \omega r)^{\frac{3}{2}}$

$$(dp + \omega dq + \omega^2 dr)^2 (p + \omega q + \omega^2 r)$$
$$- (dp + \omega^2 dq + \omega dr)^2 (p + \omega^2 q + \omega r) = 0.$$

The coefficient of $(dp)^2 + 2dqdr = (\omega - \omega^2)(q - r)$,

.... $(dq)^2 + 2drdp = (\omega - \omega^2)(r - p)$,

.... $(dr)^2 + 2dpdq = (\omega - \omega^2)(p - q)$,

$$\text{and } \frac{dp}{p} + \frac{dq}{q} + \frac{dr}{r} = 0, \quad \therefore\ -(dp)^2 = \frac{p}{q}dpdq + \frac{p}{r}dpdr;$$

$$\therefore\ \left(-\frac{p}{q}dpdq - \frac{p}{r}dpdr + 2dqdr\right)(q - r) + \ldots = 0,$$

in which the coefficient of $dqdr = 2(q - r) - \dfrac{q}{r}(r - p) - \dfrac{r}{q}(p - q)$

$$= q - r + p\left(\frac{q}{r} - \frac{r}{q}\right) = p(q - r)\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right);$$

$$\therefore\ p(q - r)dqdr + \ldots = 0 \text{ the same as } (1);$$

hence, the curve of intersection is a line of curvature.

XXI.

(1) The principal radii of curvature, at the points of the surface, $a^2x^2 = z^2(x^2 + y^2)$, where $x = y = z$, are given by the equation

$$2\rho^2 - 11\sqrt{3}a\rho + 18a^2 = 0.$$

(2) Prove that the principal curvatures are equal and opposite at points in the surface $x^2(y - z) + ayz = 0$ where it is met by the cone

$$(x^2 + 6yz)\, yz = (y - z)^4.$$

(3) A surface is generated by the revolution of a parabola about its directrix; shew that one principal radius of curvature at any point is double the other.

(4) If at any point of a surface R, R', be radii of curvature of normal sections at right angles to each other, and ρ, ρ' be principal radii of curvature, the sections corresponding to R, ρ being inclined at an angle α, prove that $\frac{\cos^2\alpha}{R} - \frac{\sin^2\alpha}{R'} = \frac{\cos^2 2\alpha}{\rho}$.

(5) If ρ, ρ' be the principal radii of curvature at any point of an ellipsoid on its line of intersection with a given concentric sphere, prove that the expression $\frac{(\rho\rho')^{\frac{1}{3}}}{\rho + \rho'}$ will be invariable.

(6) At any point of the curve of contact of a cylinder circumscribed to a surface, the product of the radius of curvature of the right section of the cylinder and the radius of curvature of the normal section of the surface, drawn through the generator of the cylinder, is equal to the product of the principal radii of curvature of the surface at the point.

(7) The normal at each point of a principal section of an ellipsoid is intersected by the normal at a consecutive point not on the principal section; shew that the locus of the point of intersection is an ellipse having four (real or imaginary) contacts with the evolute of the principal section.

(8) The points of the surface $xyz = a(yz + zx + xy)$, at which the principal curvatures are equal and opposite, lie on the cone

$$x^4(y + z) + y^4(z - x) + z^4(x - y) = 0.$$

(9) The only surface of revolution, such that the curvatures of the principal sections at every point are equal and opposite, is that produced by the revolution of a catenary about its directrix.

(10) A plane curve is wrapped upon a developable surface. If ρ be the radius of curvature of the plane curve at any point, ρ' the corresponding radius of curvature of the curve upon the surface, R the corresponding principal radius of curvature of the surface, and ϕ the angle at which the curve intersects the generator of the surface $\frac{\sin^4\phi}{R^2} = \frac{1}{\rho'^2} - \frac{1}{\rho^2}$.

(11) If θ be the inclination of any tangent to that of the principal section of least curvature, and ϕ the inclination of a section through this tangent to the corresponding normal section such that the curvature is equal to that of the principal section, ρ, ρ' being the radii of curvature of the two principal sections, prove that $2\rho \sin^2 \frac{1}{2}\phi = (\rho - \rho') \cos^2\theta$.

(12) If the product of the principal radii of curvature of a surface of revolution be constant and equal to a^2, prove that $\frac{\rho^2}{a^2} + \frac{b^2}{y^2} - 1$, where ρ is the radius of curvature of the generating curve, y the distance from the axis, and b some constant. Prove also that if the generating curve cut the axis at right angles, the surface will be a sphere.

(13) A surface is generated by the motion of a variable circle which always intersects the axis of x, and is parallel to the plane of yz. If r be the radius of the circle at a point on the axis of x, and θ the inclination of the diameter through that point to the axis of z, prove that the principal radii of curvature at the point are given by the equation $\rho^2 r + p^2 (\rho - r) = 0$, where p is the value of $\frac{dx}{d\theta}$ at the point.

(14) A surface is generated by the motion of a straight line which always intersects the axis of x, prove that the radii of curvature at any point on the axis of x are $\frac{dx}{d\phi} \frac{\cos\theta \pm 1}{\sin\theta}$, x being the distance of the point from the origin, θ the angle which the corresponding generator makes with the axis of x, and ϕ that which its projection upon the plane of yz makes with the axis of y.

(15) A surface is generated by a straight line which always intersects a given circle, and the straight line through the centre of the circle normal to its plane, prove that the principal radii of curvature of the surface, at any point on the circle, are given by the equation $\rho^2 \left(\frac{d\theta}{d\phi}\right)^2 + a\rho \cos\theta - a^2 = 0$, a being the radius of the circle, θ the angle which the generator at the point makes with the fixed line, and ϕ the angle which the radius of the circle through the point makes with a fixed radius.

(16) Two surfaces touch each other at the point P; if the principal curvatures of the first surface at P be denoted by $a \pm b$, those of the second by $a' \pm b'$; and if ω be the angle between the principal planes to which $a + b$, $a' + b'$ belong, δ the angle between the two branches at P of the curve of intersection of the surfaces, shew that

$$\cos^2\delta = \frac{a^2 - 2aa' + a'^2}{b^2 - 2bb' \cos 2\omega + b'^2}.$$

(17) If ρ, ρ' be the principal radii of curvature at a point of a surface, where the direction-cosines of the normal are l, m, n, prove that

$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{dl}{dx} + \frac{dm}{dy} + \frac{dn}{dz}.$$

(18) If a surface have contact of the second order with the conicoid

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy + 2A''x + 2B''y + 2C''z + F = 0,$$

then $$\frac{A + Cp^2 + 2B'p}{r} = \frac{Cqp + A'p + B'q + C'}{s} = \frac{B + Cq^2 + 2A'q}{t}.$$

(19) Shew that the projection, on the plane of xy, of the indicatrix at any point of the surface $z = (e^y + e^{-y})\cos x$ is a rectangular hyperbola.

(20) Shew that the indicatrix at any point of the surface $y = x \tan \frac{z}{a}$ is the part of a rectangular hyperbola which lies near the point. Prove that the section by the tangent plane near the point is the generating line and a portion of a parabola.

(21) Deduce the conditions for an umbilicus from the equation giving the radii of curvature, by making the roots of the equation equal.

(22) Shew that a sphere whose centre is the origin, and the reciprocal of whose radius is $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ touches the surface whose equation is $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{z}{c}\right)^{\frac{2}{3}} = 1$ at an umbilic.

(23) Prove that the radius of curvature of the surface $x^m + y^m + z^m = a^m$ at an umbilic is $\frac{a}{m-1} \times 3^{\frac{m-2}{2m}}$.

(24) Prove that the measure of curvature at any point of an ellipsoid is proportional to p^4, p being the perpendicular from the centre on the tangent plane.

(25) Prove that the measure of curvature at any point of the paraboloid $\frac{y^2}{b} + \frac{z^2}{c} = x$ varies as $\left(\frac{p}{z}\right)^4$, p being the perpendicular from the origin on the tangent plane.

(26) Prove that the measure of curvature at every point of the elliptic paraboloid $2z = \frac{x^2}{a} + \frac{y^2}{b}$ where it is cut by the cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to $\frac{1}{4ab}$.

(27) Shew that the specific curvature at any point on the surface $xyz = abc$, varies as the fourth power of the perpendicular from the origin on the tangent-plane at the point, and that at an umbilic it is $\frac{1}{3}(abc)^{-\frac{4}{3}}$.

(28) If a plane curve be given by the equations

$$\frac{x}{a} = \cos\theta + \log_e \tan\tfrac{1}{2}\theta, \quad \frac{y}{a} = \sin\theta,$$

the surface produced by the revolution of this curve about the axis of x will have its measure of curvature constant.

(29) In a surface, generated as in (15), if $\phi = \log \tan\frac{1}{2}\theta$, the measure of curvature will be the same at corresponding points on the fixed line and on the circle.

(30) The integral curvatures of the portions of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ cut off by the cone $\frac{x^2}{a^4} + \frac{y^2}{b^4} - \frac{z^2}{c^4} = 0$ are in the ratio of $\sqrt{2} - 1$ to $\sqrt{2} + 1$.

(31) Shew that the integral curvature of the whole surface

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 1, \text{ is } 4\pi\left\{1 - \frac{c}{\sqrt{(a^2 + c^2)}}\right\}.$$

(32) Shew that the integral curvature of the portion of a surface of revolution cut off by any plane perpendicular to the axis of revolution is $4\pi \sin^2\frac{1}{2}\alpha$, where α is the angle which the normal to the surface at any point on the curve of intersection of the plane and surface makes with the axis.

(33) If any cylinder circumscribe an ellipsoid, it divides the ellipsoid into portions whose integral curvatures are equal. Hence, if three cylinders circumscribe an ellipsoid, the integral curvature of the portion of the ellipsoid cut off is $\pi - POQ - QOR - ROP$, where O is the centre, and OP, OQ, OR are the directions of the axes of the cylinders.

(34) Prove that the integral curvature of the portion of the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, bounded by its intersection with the confocal hyperboloid of one sheet

$$\frac{x^2}{a^2 - \lambda^2} + \frac{y^2}{b^2 - \lambda^2} + \frac{z^2}{c^2 - \lambda^2} = 1 \text{ is } 2\pi\frac{c^2}{\lambda^2}\sqrt{\left\{\frac{(a^2 - \lambda^2)(b^2 - \lambda^2)}{(a^2 - c^2)(b^2 - c^2)}\right\}}.$$

(35) Find the umbilic of the surface $\frac{x^3}{a} + \frac{y^3}{b} + \frac{z^3}{c} = k^2$, and shew that, at the umbilic, $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, the directions of the three lines of curvature are given by the equations $\frac{dx}{a} = \frac{dy}{b}$, $\frac{dy}{b} = \frac{dz}{c}$, and $\frac{dz}{c} = \frac{dx}{a}$ respectively.

(36) If the inclination of two surfaces at any point of their curve of intersection be θ, s the arc of the curve of intersection, ρ_1, ρ_2 the principal radii of one surface; α the angle between the tangents to the curve and to a principal section, and ρ_1', ρ_2', α' corresponding quantities for the other surface, prove that $\frac{d\theta}{ds} = \frac{\sin 2\alpha}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) - \frac{\sin 2\alpha'}{2}\left(\frac{1}{\rho_1'} - \frac{1}{\rho_2'}\right)$. Hence, shew that if two surfaces intersect always at a constant angle, and the curve of intersection be a line of curvature of one surface, it will also be a line of curvature on the other surface.

(37) If one series of lines of curvature on a surface be plane curves, lying in parallel planes, the other series will also be plane curves.

(38) The planes drawn through the centre of an ellipsoid, parallel to the tangent planes at points along a line of curvature, envelope a cone which intersects the ellipsoid in a sphero-conic.

(39) On an umbilical conicoid, the projections of the lines of curvature on the planes of circular section, by lines parallel to an axis, form a series of confocal conics, the foci of which are the projections of the umbilics.

(40) Find the differential equation of surfaces possessing the property, that the projections, on a fixed plane, of their lines of curvature cross each other everywhere at right angles. Prove that it is satisfied by surfaces of revolution whose axes are perpendicular to the fixed plane; and obtain the general solution.

(41) Prove that the three surfaces $yz = ax$, $\sqrt{(x^2+y^2)} + \sqrt{(x^2+z^2)} = b$, $\sqrt{(x^2+y^2)} - \sqrt{(x^2+z^2)} = c$, intersect each other always at right angles; and hence prove that, on a hyperbolic paraboloid, whose principal sections are equal parabolas, the sum or the difference of the distances of any point on a line of curvature from the two generators through the vertex is constant.

(42) In the helicoid, whose equation is $y = x \tan \frac{z}{a}$, the lines of curvature are the intersections of the helicoid with the surfaces represented by the equation $\frac{2\sqrt{(x^2+y^2)}}{a} = ce^{\frac{z}{a}} + \frac{1}{c}e^{-\frac{z}{a}}$ for different values of c.

Also, prove that the principal radii of curvature are, at every point, constant, equal in magnitude, but of opposite signs.

(43) Tangent planes are drawn to a series of confocal conicoids from a fixed point on one of the axes, the locus of the points of contact is a surface; prove that three such surfaces corresponding to three points one on each axis cut one another orthogonally.

(44) Prove that the lines of curvature on the surface

$$\frac{x}{a} + \frac{y^2}{ax - a^2 + b^2} + \frac{z^2}{ax - a^2 + c^2} = 1,$$

are two systems of circles, whose planes are parallel to the axes of y and z respectively, and pass each through one of two fixed points on the axis of x.

THE END OF VOL. I.

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